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HERMITE - HADAMARD TYPE INEQUALITIES FOR FUNCTIONS WHOSE DERIVATIVES ARE (α,m) – CONVEX

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ABSTRACT

In this paper several inequalities of the right-hand side of Hermite-Hadamard inequality are obtained for the class of functions whose derivatives in absolutely value at certain powers are (α,m)-convex. Some applications to special means of positive real numbers are also given.

KEYWORDS: *(α,m)-convex functions,Hermite-Hadamard's inequality, Hölder's integral inequality*

1.INTRODUCTION

Let $f: I \subset R \to R$ be a convex function defined on the interval I of real numbers and $a, b \in I$ with $a < b$, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}$$

(1.1)

This doubly inequality is known in the literature as Hermite-Hadamard integral inequality for convex functions.

In (Miheşan, 1993), Miheşan introduced the class of (α, m)-convex functions as the following:

The function $f:[0,b] \to R$, $b > 0$, is said to be (α, m)-convex where $(\alpha, m) \in [0,1]^2$, if we have

$f\left(tx + m(1-t)y\right) \leq t^{\alpha} f(x) + m(1-t^{\alpha}) f(y)$ for all $x, y \in [0,b]$ and $t \in [0,1]$.

It can be easily that for $(\alpha, m) \in \{(0,0), (\alpha,0), (1,0), (1,m), (1,1), (\alpha,1)\}$ one obtains the following classes of functions: increasing, α-starshaped, starshaped, m-convex, convex, α-convex.

Denote by $K_m^{\alpha}(b)$ the set of all (α, m)-convex functions on $[0,b]$ for which $f(0) \leq 0$. For recent results and generalizations concerning m-convex and (α, m)-convex functions see (Bakula, et al., 2008; Bakula, et al., 2006; Ozdemir, Avcı and Kavurmacı, 2011; Ozdemir, Avcı and Set, 2010; Ozdemir, Kavurmacı and Set, 2010; Ozdemir, Set and Sarıkaya, 2011; Sarıkaya et al., 2010; Sardari et al. 2009).

In (Dragomir and Agarwal, 1998.) Dragomir and Agarwal established the following result connected with the right-hand side of (1.1).

Theorem 1.1 Let $f: I \subset R \to R$ be a differentiable mapping on I°, where $a, b \in I$ with $a < b$. If $|f'|$ is convex on $[a,b]$, then the following inequality holds:

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{8}\left[|f'(a)| + |f'(b)|\right].$$

(1.2)

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In (Set et al., 2009) the following inequality of Hermite-Hadamard type for (α,m)-convex functions holds:

Theorem 1.2. Let $f:[0,\infty)\to R$ be an (α,m)-convex function with $(\alpha,m)\in(0,1]^2$. If $0\le a<b<\infty$ and $f\in L[a,b]$, then one has the inequality:

$$\frac{1}{b-a}\int_a^b f(x)dx \le \min\left\{\frac{f(a)+\alpha m f\left(\frac{b}{m}\right)}{\alpha+1},\frac{f(b)+\alpha m f\left(\frac{a}{m}\right)}{\alpha+1}\right\}. \quad (1.3)$$

In (Bakula et al., 2008) the following Hermite-Hadamard type inequalities for $m-$ and (α,m)-convex functions were obtained.

Theorem 1.3. Let I be an open real interval such that $[0,\infty)\subset I$. Let $f:I\to R$ be a differentiable mapping on I such that $f'\in L[a,b]$, where $0\le a<b<\infty$. If $|f'|^q$ is m-convex on $[a,b]$, for some fixed $m\in(0,1]$ and $q\in(1,\infty)$, then

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{b-a}{4}\left(\frac{q-1}{2q-1}\right)^{\frac{q-1}{q}}\left(\mu_1^{\frac{1}{q}}+\mu_2^{\frac{1}{q}}\right) \quad (1.4)$$

$$\le \frac{b-a}{4}\left(\mu_1^{\frac{1}{q}}+\mu_2^{\frac{1}{q}}\right)$$

where

$$\mu_1=\min\left\{\frac{|f'(a)|^q+m\left|f'\left(\frac{a+b}{2m}\right)\right|^q}{2},\frac{\left|f'\left(\frac{a+b}{2}\right)\right|^q+m\left|f'\left(\frac{a}{m}\right)\right|^q}{2}\right\},$$

$$\mu_2=\min\left\{\frac{|f'(b)|^q+m\left|f'\left(\frac{a+b}{2m}\right)\right|^q}{2},\frac{\left|f'\left(\frac{a+b}{2}\right)\right|^q+m\left|f'\left(\frac{b}{m}\right)\right|^q}{2}\right\}.$$

Theorem 1.4. Let I be an open real interval such that $[0,\infty)\subset I$. Let $f:I\to R$ be a differentiable mapping on I such that $f'\in L[a,b]$, where $0\le a<b<\infty$. If $|f'|^q$ is (α,m)-convex on $[a,b]$, for some fixed $\alpha,m\in(0,1]$ and $q\in[1,\infty)$, then

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \quad (1.5)$$

$$\le \frac{b-a}{2}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}$$

$$\times\min\left\{\left(v_1|f'(a)|^q+mv_2\left|f'\left(\frac{b}{m}\right)\right|^q\right)^{\frac{1}{q}},\left(v_1|f'(b)|^q+mv_2\left|f'\left(\frac{a}{m}\right)\right|^q\right)^{\frac{1}{q}}\right\}$$

where

$$v_1=\frac{1}{(\alpha+1)(\alpha+2)}\left[\alpha+\left(\frac{1}{2}\right)^{\alpha}\right]$$

and

$$v_2=\frac{1}{(\alpha+1)(\alpha+2)}\left[\frac{\alpha^2+\alpha+2}{2}-\left(\frac{1}{2}\right)^{\alpha}\right].$$

The main aim of this paper is to establish new inequalities of Hermite-Hadamard type for the class of functions whose derivatives in absolutely value at certain powers are (α,m)-convex.

2. INEQUALITIES FOR FUNCTIONS WHOSE DERIVATIVES ARE $(\alpha,m)-$CONVEX

In order to prove our main resuls we need the following lemma:

Lemma 2.1 Let $f:I\subset R\to R$ be a differentiable mapping on I°, $a,b\in I$ with $a<b$. If $f'\in L[a,b]$ and $\lambda,\mu\in[0,\infty)$, $\lambda+\mu>0$, then the following equality holds:

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$$\frac{\lambda f(a)+\mu f(b)}{\lambda+\mu}-\frac{1}{b-a}\int_a^b f(x)dx$$
$$=\frac{b-a}{\lambda+\mu}\int_0^1 [(\lambda+\mu)t-\lambda]\, f'(tb+(1-t)a)dt$$

Proof. Integration by parts we have

$$I=\int_0^1[(\lambda+\mu)t-\lambda]\, f'(tb+(1-t)a)dt$$
$$=[(\lambda+\mu)t-\lambda]\frac{f(tb+(1-t)a)}{b-a}\Big|_0^1$$
$$-\frac{\lambda+\mu}{b-a}\int_0^1 f(tb+(1-t)a)dt$$
$$=\frac{\lambda f(a)+\mu f(b)}{b-a}-\frac{\lambda+\mu}{b-a}\int_0^1 f(tb+(1-t)a)dt.$$

Setting $x=tb+(1-t)a$, and $dx=(b-a)dt$ gives

$$I=\frac{\lambda f(a)+\mu f(b)}{b-a}-\frac{\lambda+\mu}{(b-a)^2}\int_a^b f(x)dx.$$

Therefore,

$$\left(\frac{b-a}{\lambda+\mu}\right)I=\frac{\lambda f(a)+\mu f(b)}{\lambda+\mu}-\frac{1}{b-a}\int_a^b f(x)dx$$

which completes the proof. □

The next theorem gives a new refinement of the upper Hermite-Hadamard inequality for (α,m)-convex functions.

Theorem 2.2. Let $f:I\subset[0,\infty)\to R$ be a differentiable mapping on I° such that $f'\in L[a,b]$, where $a,b\in I^\circ$ with $a<b$. If $|f'|^q$ is (α,m)-convex on $[a,b]$, for some fixed $(\alpha,m)\in(0,1]^2,\lambda,\mu\in[0,\infty)$ with $\lambda+\mu>0$, and $q\geq 1$, then the following inequality holds:

$$\left|\frac{\lambda f(a)+\mu f(b)}{\lambda+\mu}-\frac{1}{b-a}\int_a^b f(x)dx\right| \tag{2.1}$$
$$\leq\frac{b-a}{\lambda+\mu}\left(\frac{\lambda^2+\mu^2}{2(\lambda+\mu)}\right)^{\frac{q-1}{q}}$$
$$\times\min\left\{\left(\gamma_1\left|f'(b)\right|^q+m\gamma_2\left|f'(\frac{a}{m})\right|^q\right)^{\frac{1}{q}},\left(\gamma_3\left|f'(a)\right|^q+m\gamma_4\left|f'(\frac{b}{m})\right|^q\right)^{\frac{1}{q}}\right\}$$

where

$$\gamma_1=\frac{1}{(\alpha+1)(\alpha+2)}\left[\frac{2\lambda^{\alpha+2}}{(\lambda+\mu)^{\alpha+1}}+(\alpha+1)\mu-\lambda\right],$$
$$\gamma_2=\frac{\lambda^2+\mu^2}{2(\lambda+\mu)}-\gamma_1,$$

and

$$\gamma_3=\frac{1}{(\alpha+1)(\alpha+2)}\left[\frac{2\mu^{\alpha+2}}{(\lambda+\mu)^{\alpha+1}}+(\alpha+1)\lambda-\mu\right],$$
$$\gamma_4=\frac{\lambda^2+\mu^2}{2(\lambda+\mu)}-\gamma_3.$$

Proof. Suppose that $q=1$. From Lemma 2.1 and using the (α,m)-convexity of $|f'|$, we have

$$\left|\frac{\lambda f(a)+\mu f(b)}{\lambda+\mu}-\frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\leq\frac{b-a}{\lambda+\mu}\int_0^1\left|(\lambda+\mu)t-\lambda\right|\left|f'(tb+(1-t)a)\right|dt$$
$$\leq\frac{b-a}{\lambda+\mu}\int_0^1\left|(\lambda+\mu)t-\lambda\right|$$
$$\times\left[t^\alpha\left|f'(b)\right|+m(1-t^\alpha)\left|f'(\frac{a}{m})\right|\right]dt$$

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$$= \frac{b-a}{\lambda+\mu}\int_0^1 |(\lambda+\mu)t-\lambda| t^{\alpha} \left| f'(b) \right|$$

$$+ m(1-t^{\alpha})|(\lambda+\mu)t-\lambda| \left| f'(\frac{a}{m}) \right| dt$$

where

$$\int_0^1 |(\lambda+\mu)t-\lambda| t^{\alpha} dt$$

$$= \int_0^{\frac{\lambda}{\lambda+\mu}} \left[\lambda-(\lambda+\mu)t\right] t^{\alpha} dt$$

$$+ \int_{\frac{\lambda}{\lambda+\mu}}^1 \left[(\lambda+\mu)t-\lambda\right] t^{\alpha} dt$$

$$= \frac{1}{(\alpha+1)(\alpha+2)}\left[\frac{2\lambda^{\alpha+2}}{(\lambda+\mu)^{\alpha+1}} + (\alpha+1)\mu-\lambda\right]$$

$$= \gamma_1$$

(2.2)

and

$$\int_0^1 |(\lambda+\mu)t-\lambda| \left(1-t^{\alpha}\right) dt$$

$$= \frac{\lambda^2+\mu^2}{2(\lambda+\mu)} - \gamma_1 = \gamma_2, \qquad (2.3)$$

hence

$$\left| \frac{\lambda f(a)+\mu f(b)}{\lambda+\mu} - \frac{1}{b-a}\int_a^b f(x)dx \right|$$

$$\le \frac{b-a}{\lambda+\mu}\left(\gamma_1 \left| f'(b) \right| + m\gamma_2 \left| f'(\frac{a}{m}) \right| \right).$$

Since

$$\int_0^1 |(\lambda+\mu)t-\lambda| \left| f'(tb+(1-t)a) \right| dt$$

$$= \int_0^1 |(\lambda+\mu)t-\mu| \left| f'(ta+(1-t)b) \right| dt$$

Analogously we obtain

$$\left| \frac{\lambda f(a)+\mu f(b)}{\lambda+\mu} - \frac{1}{b-a}\int_a^b f(x)dx \right|$$

$$\le \frac{b-a}{\lambda+\mu}\left(\gamma_3 \left| f'(a) \right| + m\gamma_4 \left| f'(\frac{b}{m}) \right| \right),$$

where

$$\gamma_3 = \frac{1}{(\alpha+1)(\alpha+2)}\left[\frac{2\lambda^{\alpha+2}}{(\lambda+\mu)^{\alpha+1}} + (\alpha+1)\lambda-\mu\right]$$

and

$$\gamma_4 = \frac{\lambda^2+\mu^2}{2(\lambda+\mu)} - \gamma_3$$

which copmletes the proof for this case.

Suppose now that $q\in(1,\infty)$. From Lemma 2.1 and using the Hölder's integral inequality, we have

$$\int_0^1 |(\lambda+\mu)t-\lambda| \left| f'(tb+(1-t)a) \right| dt \qquad (2.4)$$

$$\le \frac{b-a}{\lambda+\mu}\left(\int_0^1 |(\lambda+\mu)t-\lambda| dt \right)^{\frac{q-1}{q}}$$

$$\times \left(\int_0^1 |(\lambda+\mu)t-\lambda| \left| f'(tb+(1-t)a) \right|^q dt \right)^{\frac{1}{q}}$$

Since $\left| f' \right|^q$ is (α,m)-convex on $[a,b]$, we know that for every $t\in[0,1]$

$$\left| f'(tb+m(1-t)\frac{a}{m}) \right|^q$$

$$\le t^{\alpha} \left| f'(b) \right|^q + m(1-t^{\alpha}) \left| f'(\frac{a}{m}) \right|^q. \qquad (2.5)$$

From (2.2), (2.3), (2.4) and (2.5), we have

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$$\left|\frac{\lambda f(a)+\mu f(b)}{\lambda+\mu}-\frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\leq\frac{b-a}{\lambda+\mu}\left(\frac{\lambda^2+\mu^2}{2(\lambda+\mu)}\right)^{\frac{q-1}{q}}\left(\gamma_1\left|f'(b)\right|^q+m\gamma_2\left|f'(\frac{a}{m})\right|^q\right)^{\frac{1}{q}}$$

and analogously

$$\left|\frac{\lambda f(a)+\mu f(b)}{\lambda+\mu}-\frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\leq\frac{b-a}{\lambda+\mu}\left(\frac{\lambda^2+\mu^2}{2(\lambda+\mu)}\right)^{\frac{q-1}{q}}\left(\gamma_3\left|f'(a)\right|^q+m\gamma_4\left|f'(\frac{b}{m})\right|^q\right)^{\frac{1}{q}}$$

which completes the proof. □

Corollary 2.3. Suppose that all the assumptions of Theorem 2.2 are satisfied,

i) In the inequality (2.1), If we choose $\lambda=\mu$, we obtain the inequality in (1.5).
ii) In the inequality (2.1) If we choose $\lambda=\mu,\quad m=1,\ q=1$ and $\alpha=1$ we obtain the inequality in (1.2).
iii) In the inequality (2.1) If we choose $m=\alpha=1$ we have

$$\left|\frac{\lambda f(a)+\mu f(b)}{\lambda+\mu}-\frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\leq\frac{b-a}{\lambda+\mu}\left(\frac{\lambda^2+\mu^2}{2(\lambda+\mu)}\right)^{\frac{q-1}{q}}$$
$$\times\min\left\{\left(\gamma_1\left|f'(b)\right|^q+\gamma_2\left|f'(a)\right|^q\right)^{\frac{1}{q}},\right.$$
$$\left.\left(\gamma_3\left|f'(a)\right|^q+\gamma_4\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right\} \tag{2.6}$$

where

$$\gamma_1=\frac{1}{6}\left[\frac{2\lambda^3}{(\lambda+\mu)^2}+2\mu-\lambda\right],\ \gamma_2=\frac{\lambda^2+\mu^2}{2(\lambda+\mu)}-\gamma_1,$$

and

$$\gamma_3=\frac{1}{6}\left[\frac{2\mu^3}{(\lambda+\mu)^2}+2\lambda-\mu\right],\ \gamma_4=\frac{\lambda^2+\mu^2}{2(\lambda+\mu)}-\gamma_3.$$

Theorem 2.4. Let $f:I\subset[0,\infty)\to R$ be a differentiable mapping on I° such that $f'\in L[a,b]$, where $a,b\in I^\circ$ with $a<b$. If $\left|f'\right|^q$ is (α,m)-convex on $[a,b]$, for some fixed $(\alpha,m)\in(0,1]^2,\lambda,\mu\in[0,\infty)$ with $\lambda+\mu>0$, and $q>1$, then the following inequality holds:

$$\left|\frac{\lambda f(a)+\mu f(b)}{\lambda+\mu}-\frac{1}{b-a}\int_a^b f(x)dx\right|\leq\frac{b-a}{(\lambda+\mu)^2} \tag{2.7}$$
$$\times\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\lambda^2M_1^{\frac{1}{q}}+\mu^2M_2^{\frac{1}{q}}\right]$$

where

$$M_1=\min\left\{\frac{\left|f'(a)\right|^q+\alpha m\left|f'\left(\frac{\lambda b+\mu a}{m(\lambda+\mu)}\right)\right|^q}{\alpha+1},\frac{\left|f'(\frac{\lambda b+\mu a}{\lambda+\mu})\right|^q+\alpha m\left|f'\left(\frac{a}{m}\right)\right|^q}{\alpha+1}\right\},$$
$$M_2=\min\left\{\frac{\left|f'(b)\right|^q+\alpha m\left|f'\left(\frac{\lambda b+\mu a}{m(\lambda+\mu)}\right)\right|^q}{\alpha+1},\frac{\left|f'(\frac{\lambda b+\mu a}{\lambda+\mu})\right|^q+\alpha m\left|f'\left(\frac{b}{m}\right)\right|^q}{\alpha+1}\right\}$$

and $\frac{1}{p}+\frac{1}{q}=1.$

Proof. From Lemma 2.1 and using the Hölder inequality, we have

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$$\left|\frac{\lambda f(a)+\mu f(b)}{\lambda+\mu}-\frac{1}{b-a}\int_a^b f(x)dx\right|$$

$$\leq\frac{b-a}{\lambda+\mu}\left(\int_0^{\frac{\lambda}{\lambda+\mu}}[\lambda-(\lambda+\mu)t]^p dt\right)^{\frac{1}{p}}$$

$$\times\left(\int_0^{\frac{\lambda}{\lambda+\mu}}\left|f'(tb+(1-t)a)\right|^q dt\right)^{\frac{1}{q}}$$

$$+\frac{b-a}{\lambda+\mu}\left(\int_{\frac{\lambda}{\lambda+\mu}}^1[(\lambda+\mu)t-\lambda]^p dt\right)^{\frac{1}{p}}$$

$$\times\left(\int_{\frac{\lambda}{\lambda+\mu}}^1\left|f'(tb+(1-t)a)\right|^q dt\right)^{\frac{1}{q}}$$

$$\leq\frac{b-a}{\lambda+\mu}\left[\left(\frac{\lambda^{p+1}}{(p+1)(\lambda+\mu)}\right)^{\frac{1}{p}}\left(\frac{\lambda}{\lambda+\mu}M_1\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\frac{\mu^{p+1}}{(p+1)(\lambda+\mu)}\right)^{\frac{1}{p}}\left(\frac{\mu}{\lambda+\mu}M_2\right)^{\frac{1}{q}}\right]$$

$$=\frac{b-a}{\lambda+\mu}\times\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\frac{\lambda^2}{\lambda+\mu}M_1^{\frac{1}{q}}+\frac{\mu^2}{\lambda+\mu}M_2^{\frac{1}{q}}\right]$$

where we use the fact that

$$\int_0^{\frac{\lambda}{\lambda+\mu}}[\lambda-(\lambda+\mu)t]^p dt=\frac{\lambda^{p+1}}{(p+1)(\lambda+\mu)}$$

$$\int_{\frac{\lambda}{\lambda+\mu}}^1[(\lambda+\mu)t-\lambda]^p dt=\frac{\mu^{p+1}}{(p+1)(\lambda+\mu)}$$

and by Theorem 1.2 we get

$$\frac{\lambda+\mu}{\lambda}\int_0^{\frac{\lambda}{\lambda+\mu}}\left|f'(tb+(1-t)a)\right|^q dt$$

$$=\frac{1}{\frac{\lambda}{\lambda+\mu}(b-a)}\int_a^{\frac{\lambda b+\mu a}{\lambda+\mu}}\left|f'(x)\right|^q dx$$

$$\leq\min\left\{\frac{\left|f'(a)\right|^q+\alpha m\left|f'\left(\frac{\lambda b+\mu a}{m(\lambda+\mu)}\right)\right|^q}{\alpha+1},\right.$$

$$\left.\frac{\left|f'\left(\frac{\lambda b+\mu a}{\lambda+\mu}\right)\right|^q+\alpha m\left|f'\left(\frac{a}{m}\right)\right|^q}{\alpha+1}\right\},$$

$$\frac{\lambda+\mu}{\mu}\int_{\frac{\lambda}{\lambda+\mu}}^1\left|f'(tb+(1-t)a)\right|^q dt$$

$$=\frac{1}{\frac{\mu}{\lambda+\mu}(b-a)}\int_{\frac{\lambda b+\mu a}{\lambda+\mu}}^b\left|f'(x)\right|^q dx$$

$$\leq\min\left\{\frac{\left|f'(b)\right|^q+\alpha m\left|f'\left(\frac{\lambda b+\mu a}{m(\lambda+\mu)}\right)\right|^q}{\alpha+1},\right.$$

$$\left.\frac{\left|f'\left(\frac{\lambda b+\mu a}{\lambda+\mu}\right)\right|^q+\alpha m\left|f'\left(\frac{b}{m}\right)\right|^q}{\alpha+1}\right\}.$$

which completes the proof. □

Corollary 2.5. Suppose that all the assumptions of Theorem 2.4 are satisfied, in this case:

i) In the inequality (2.7) if we choose $\lambda=\mu$ and $\alpha=1$ we obtain the inequality in (1.4).

ii) In the inequality (2.7) if we choose $m=\alpha=1$ we have

$$\left|\frac{\lambda f(a)+\mu f(b)}{\lambda+\mu}-\frac{1}{b-a}\int_a^b f(x)dx\right| \tag{2.8}$$

$$\leq\frac{b-a}{(\lambda+\mu)^2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\lambda^2 M_1^{\frac{1}{q}}+\mu^2 M_2^{\frac{1}{q}}\right]$$

where

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$$M_1=\frac{\left|f'(a)\right|^q+\left|f'\left(\frac{\lambda b+\mu a}{\lambda+\mu}\right)\right|^q}{2},$$

$$M_2=\frac{\left|f'(b)\right|^q+\left|f'\left(\frac{\lambda b+\mu a}{\lambda+\mu}\right)\right|^q}{2}.$$

Theorem 2.6. Let $f:I\subset[0,\infty)\to R$ be a differentiable mapping on I° such that $f'\in L[a,b]$, where $a,b\in I^\circ$ with $a<b$. If $\left|f'\right|^q$ is (α,m)-convex on $[a,b]$, for some fixed $(\alpha,m)\in(0,1]^2,\lambda,\mu\in[0,\infty)$ with $\lambda+\mu>0$, and $q>1$, then the following inequality holds:

$$\left|\frac{\lambda f(a)+\mu f(b)}{\lambda+\mu}-\frac{1}{b-a}\int_a^b f(x)dx\right|$$

$$\le\frac{b-a}{\lambda+\mu}\left(\frac{\lambda^{p+1}+\mu^{p+1}}{(p+1)(\lambda+\mu)}\right)^{\frac{1}{p}}\left(\frac{1}{\alpha+1}\right)^{\frac{1}{q}}\min\left\{K_1^{\frac{1}{q}},K_2^{\frac{1}{q}}\right\},$$

where

$$K_1=\left|f'(b)\right|^q+m\alpha\left|f'\left(\frac{a}{m}\right)\right|^q$$

$$K_2=\left|f'(a)\right|^q+m\alpha\left|f'\left(\frac{b}{m}\right)\right|^q$$

and $\frac{1}{p}+\frac{1}{q}=1.$

Proof. From Lemma 2.1 and using the Hölder's integral inequality, we have

$$\left|\frac{\lambda f(a)+\mu f(b)}{\lambda+\mu}-\frac{1}{b-a}\int_a^b f(x)dx\right|$$

$$\le\int_0^1\left|(\lambda+\mu)t-\lambda\right|\left|f'(tb+(1-t)a)\right|dt$$

$$\le\frac{b-a}{\lambda+\mu}\left(\int_0^1\left|(\lambda+\mu)t-\lambda\right|^p dt\right)^{\frac{1}{p}}$$

$$\times\left(\int_0^1\left|f'(tb+(1-t)a)\right|^q dt\right)^{\frac{1}{q}}$$

$$\le\frac{b-a}{\lambda+\mu}\left(\frac{\lambda^{p+1}+\mu^{p+1}}{(p+1)(\lambda+\mu)}\right)^{\frac{1}{p}}$$

$$\times\left(\int_0^1 t^\alpha\left|f'(b)\right|^q+m(1-t^\alpha)\left|f'\left(\frac{a}{m}\right)\right|^q dt\right)^{\frac{1}{q}}$$

$$=\frac{b-a}{\lambda+\mu}\left(\frac{\lambda^{p+1}+\mu^{p+1}}{(p+1)(\lambda+\mu)}\right)^{\frac{1}{p}}\left(\frac{1}{\alpha+1}\right)^{\frac{1}{q}}$$

$$\times\left(\left|f'(b)\right|^q+m\alpha\left|f'\left(\frac{a}{m}\right)\right|^q\right)^{\frac{1}{q}}.$$

Analogously we obtain

$$\left|\frac{\lambda f(a)+\mu f(b)}{\lambda+\mu}-\frac{1}{b-a}\int_a^b f(x)dx\right|\le\frac{b-a}{\lambda+\mu}\left(\frac{1}{\alpha+1}\right)^{\frac{1}{q}}\tag{2.9}$$

$$\times\left(\frac{\lambda^{p+1}+\mu^{p+1}}{(p+1)(\lambda+\mu)}\right)^{\frac{1}{p}}\left(\left|f'(a)\right|^q+m\alpha\left|f'\left(\frac{b}{m}\right)\right|^q\right)^{\frac{1}{q}}.$$

□

Corollary 2.7. Suppose that all the assumptions of Theorem 2.6 are satisfied, in this case:

i) In the inequality (2.9) if we choose $\lambda=\mu$, then the following inequality holds:

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|$$

$$\le\frac{b-a}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{1}{\alpha+1}\right)^{\frac{1}{q}}\min\left\{K_1^{\frac{1}{q}},K_2^{\frac{1}{q}}\right\},$$

ii) In the inequality (2.9) if we choose $m=\alpha=1$, we have

$$\left|\frac{\lambda f(a)+\mu f(b)}{\lambda+\mu}-\frac{1}{b-a}\int_a^b f(x)dx\right|\le\frac{b-a}{\lambda+\mu}$$

$$\times\left(\frac{\lambda^{p+1}+\mu^{p+1}}{\lambda+\mu}\right)^{\frac{1}{p}}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{1}{2}\right)^{\frac{1}{q}}\left(\left|f'(a)\right|^q+\left|f'(b)\right|^q\right)^{\frac{1}{q}}$$

(2.10)

3. SOME APPLICATIONS FOR SPECIAL MEANS

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Let us recall the following special means of two nonnegative number a, b with $b \geq a$ and $\alpha \in [0,1]$:

i) The weighted arithmetic mean

$$A_\alpha(a,b) := \alpha a + (1-\alpha)b,\ a,b \geq 0.$$

ii) The unweighted arithmetic mean

$$A(a,b) := \frac{a+b}{2},\ a,b \geq 0.$$

iii) The weighted harmonic mean

$$H_\alpha(a,b) := \left(\frac{\alpha}{a} + \frac{1-\alpha}{b}\right)^{-1},\ a,b > 0.$$

iv) The unweighted harmonic mean

$$H(a,b) := \frac{2ab}{a+b},\ a,b > 0.$$

v) The Logarithmic mean

$$L(a,b) := \begin{cases} \dfrac{b-a}{\ln b - \ln a} & if\ a \neq b \\ b & if\ a = b \end{cases},\ a,b > 0.$$

vi) The p-Logarithmic mean

$$L_p(a,b) := \begin{cases} \left(\dfrac{b^{p+1} - a^{p+1}}{(p+1)(b-a)}\right)^{\frac{1}{p}} & if\ a \neq b \\ b & if\ a = b \end{cases},$$

$a,b > 0,\ p \in Z \setminus \{-1, 0\}$.

Proposition 3.1 Let $a, b \in R$ with $0 < a < b$ and $n \in Z, |n| \geq 2.$ Then, we have the following inequality:

$$\left|A_{\frac{\lambda}{\lambda+\mu}}(a^n, b^n) - L_n^n(a,b)\right| \leq \frac{b-a}{\lambda+\mu}\left(\frac{\lambda^2+\mu^2}{2(\lambda+\mu)}\right)^{\frac{q-1}{q}}$$

$$\times |n| \min\left\{\left(\gamma_1 b^{q(n-1)} + \gamma_2 a^{q(n-1)}\right)^{\frac{1}{q}}, \left(\gamma_3 a^{q(n-1)} + \gamma_4 b^{q(n-1)}\right)^{\frac{1}{q}}\right\}$$

where

$$\gamma_1 = \frac{1}{6}\left[\frac{2\lambda^3}{(\lambda+\mu)^2} + 2\mu - \lambda\right], \gamma_2 = \frac{\lambda^2+\mu^2}{2(\lambda+\mu)} - \gamma_1,$$

$$\gamma_3 = \frac{1}{6}\left[\frac{2\mu^3}{(\lambda+\mu)^2} + 2\lambda - \mu\right], \gamma_4 = \frac{\lambda^2+\mu^2}{2(\lambda+\mu)} - \gamma_3,$$

$\lambda, \mu \in [0, \infty)$ with $\lambda + \mu > 0$ and $q \geq 1$.

Proof. The assertion follows from inequality (2.6) in Corollary 2.3, for $f : (0,\infty) \to R,\ f(x) = x^n$ □

Proposition 3.2. Let $a, b \in R$ with $0 < a < b$ and $n \in Z, |n| \geq 2.$ Then, we have the following inequality:

$$\left|A_{\frac{\lambda}{\lambda+\mu}}(a^n, b^n) - L_n^n(a,b)\right|$$

$$\leq \frac{b-a}{(\lambda+\mu)^2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}} |n| \left[\lambda^2 M_1^{\frac{1}{q}} + \mu^2 M_2^{\frac{1}{q}}\right]$$

where

$$M_1 = A\left(a^{(n-1)q}, A_{\frac{\lambda}{\lambda+\mu}}^{(n-1)q}(b,a)\right),$$

$$M_2 = A\left(b^{(n-1)q}, A_{\frac{\lambda}{\lambda+\mu}}^{(n-1)q}(b,a)\right),$$

$\lambda, \mu \in [0, \infty)$ with $\lambda + \mu > 0$, and $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$.

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Proof. The assertion follows from inequality (2.8) in Corollary 2.5, for $f:(0,\infty)\to R,\ f(x)=x^n$ □

Proposition 3.3. Let $a,b\in R$ with $0<a<b$ and $n\in Z, |n|\geq 2$. Then, we have the following inequality:

$$\left|A_{\frac{\lambda}{\lambda+\mu}}\left(a^n,b^n\right)-L_n^n(a,b)\right|$$

$$\leq \frac{b-a}{\lambda+\mu}\left(\frac{\lambda^{p+1}+\mu^{p+1}}{\lambda+\mu}\right)^{\frac{1}{p}}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}|n|A^{\frac{1}{q}}\left(a^{(n-1)q},b^{(n-1)q}\right)$$

$\lambda,\mu\in[0,\infty)$ with $\lambda+\mu>0$, and $q>1$ with $\frac{1}{p}+\frac{1}{q}=1.$

Proof. The assertion follows from inequality (2.10) in Corollary 2.7, for $f:(0,\infty)\to R,\ f(x)=\frac{1}{x}$. □

Proposition 3.4. Let $a,b\in R$ with $0<a<b$. Then we have the following inequality:

$$\left|H^{-1}_{\frac{\lambda}{\lambda+\mu}}(a,b)-L^{-1}(a,b)\right|$$

$$\leq \frac{b-a}{\lambda+\mu}\left(\frac{\lambda^2+\mu^2}{2(\lambda+\mu)}\right)^{\frac{q-1}{q}}$$

$$\times\min\left\{\left(\gamma_1\frac{1}{b^{2q}}+\gamma_2\frac{1}{a^{2q}}\right)^{\frac{1}{q}},\left(\gamma_3\frac{1}{a^{2q}}+\gamma_4\frac{1}{b^{2q}}\right)^{\frac{1}{q}}\right\}$$

where

$$\gamma_1=\frac{1}{6}\left[\frac{2\lambda^3}{(\lambda+\mu)^2}+2\mu-\lambda\right],\gamma_2=\frac{\lambda^2+\mu^2}{2(\lambda+\mu)}-\gamma_1,$$

$$\gamma_3=\frac{1}{6}\left[\frac{2\mu^3}{(\lambda+\mu)^2}+2\lambda-\mu\right],\gamma_4=\frac{\lambda^2+\mu^2}{2(\lambda+\mu)}-\gamma_3,$$

$\lambda,\mu\in[0,\infty)$ with $\lambda+\mu>0$ and $q\geq 1$.

Proof. The assertion follows from inequality (2.6) in Corollary 2.3, for $f:(0,\infty)\to R,\ f(x)=x^n$ □

Proposition 3.5. Let $a,b\in R$ with $0<a<b.$ Then we have the following inequality:

$$\left|H^{-1}_{\frac{\lambda}{\lambda+\mu}}(a,b)-L^{-1}(a,b)\right|$$

$$\leq\frac{b-a}{(\lambda+\mu)^2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\lambda^2M_1^{\frac{1}{q}}+\mu^2M_2^{\frac{1}{q}}\right]$$

where

$$M_1=H^{-1}\left(a^{2q},A^{2q}_{\frac{\lambda}{\lambda+\mu}}(b,a)\right),$$

$$M_2=H^{-1}\left(b^{2q},A^{2q}_{\frac{\lambda}{\lambda+\mu}}(b,a)\right)$$

$\lambda,\mu\in[0,\infty)$ with $\lambda+\mu>0$, and $q>1$ with $\frac{1}{p}+\frac{1}{q}=1.$

Proof. The assertion follows from inequality (2.8) in Corollary 2.5, for $f:(0,\infty)\to R,\ f(x)=\frac{1}{x}$ □

Proposition 3.6. Let $a,b\in R$ with $0<a<b$. Then we have the following inequality:

$$\left|H^{-1}_{\frac{\lambda}{\lambda+\mu}}(a,b)-L^{-1}(a,b)\right|\leq\frac{b-a}{\lambda+\mu}$$

$$\times A^{\frac{1}{p}}_{\frac{\lambda}{\lambda+\mu}}\left(\lambda^p,\mu^p\right)\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{1}{2}\right)^{\frac{1}{q}}H^{\frac{-1}{q}}_{\frac{1}{2}}\left(a^{2q},b^{2q}\right),$$

$\lambda,\mu\in[0,\infty)$ with $\lambda+\mu>0$, and $q>1$ with $\frac{1}{p}+\frac{1}{q}=1.$

Proof. The assertion follows from inequality (2.10) in Corollary 2.7, for $f:(0,\infty)\to R,\ f(x)=\frac{1}{x}$ □

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