HERMITE - HADAMARD TYPE INEQUALITIES FOR FUNCTIONS WHOSE DERIVATIVES ARE \((\alpha,m)\) – CONVEX

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ABSTRACT

In this paper several inequalities of the right-hand side of Hermite-Hadamard inequality are obtained for the class of functions whose derivatives in absolutely value at certain powers are \((\alpha,m)\)-convex. Some applications to special means of positive real numbers are also given.

KEYWORDS: \((\alpha,m)\)-convex functions, Hermite-Hadamard's inequality, Hölder's integral inequality

1. INTRODUCTION

Let \(f : I \subset \mathbb{R} \to \mathbb{R}\) be a convex function defined on the interval \(I\) of real numbers and \(a,b \in I\) with \(a \leq b\), then

\[
f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a) + f(b)}{2}\]

(1.1)

This doubly inequality is known in the literature as Hermite-Hadamard integral inequality for convex functions.

In (Miheşan, 1993), Miheşan introduced the class of \((\alpha,m)\)-convex functions as the following: The function \(f : [0,b] \to \mathbb{R}, b > 0\), is said to be \((\alpha,m)\)-convex where \((\alpha,m) \in [0,1]^2\), if we have

\[
f\left(tx + m(1-t)y\right) \leq tf(x) + m(1-t)f(y) \quad \text{for all} \quad x, y \in [0,b] \quad \text{and} \quad t \in [0,1].
\]

It can be easily that for \((\alpha,m) \in \{(0,0),(\alpha,0),(1,0),(1,m),(1,1),(\alpha,1)\}\) one obtains the following classes of functions: increasing, \(\alpha\) -starshaped, starshaped, \(m\) -convex, convex, \(\alpha\) -convex.

Denote by \(K_m^\alpha(b)\) the set of all \((\alpha,m)\)-convex functions on \([0,b]\) for which \(f(0) = 0\). For recent results and generalizations concerning \(m\) -convex and \((\alpha,m)\)-convex functions see (Bakula, et al., 2008; Bakula, et al., 2006; Ozdemir, Avci and Kavurmaci, 2011; Ozdemir, Avci and Set, 2010; Ozdemir, Kavurmaci and Set, 2010; Ozdemir, Set and Sarıkaya, 2011; Sarıkaya et al., 2010, Sardari et al. 2009).

In (Dragomir and Agarwal, 1998.) Dragomir and Agarwal established the following result connected with the right-hand side of (1.1).

Theorem 1.1 Let \(f : I \subset \mathbb{R} \to \mathbb{R}\) be a differentiable mapping on \(I\), where \(a,b \in I\) with \(a < b\). If \(|f'|\) is convex on \([a,b]\), then the following inequality holds:

\[
\left|\frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx\right| \leq \frac{b-a}{8}\left[f'(a) + f'(b)\right].
\]

(1.2)
In (Set et al., 2009) the following inequality of Hermite-Hadamard type for \((α,m)\)-convex functions holds:

**Theorem 1.2.** Let \( f : [0,∞) \to R \) be an \((α,m)\)-convex function with \((α,m) ∈ (0,1]^2\). If
\[
0 ≤ a < b < ∞ \text{ and } f ∈ L[a, b],
\]
then one has the inequality:
\[
\frac{1}{b-a} \int_a^b f(x)dx ≤ \min \left\{ \frac{f(a) + αmf(\frac{b}{m})}{α+1} , \frac{f(b) + αmf(\frac{a}{m})}{α+1} \right\}.
\]
(1.3)

In (Bakula et al., 2008) the following Hermite-Hadamard type inequalities for \(m-\) and \((α,m)\)-convex functions were obtained.

**Theorem 1.3.** Let \( I \) be an open real interval such that \([0,∞) ⊂ I\). Let \( f : I \to R \) be a differentiable mapping on \( I \) such that \( f' ∈ L[a, b] \), where \(0 ≤ a < b < ∞\). If \( |f'|^q \) is \( m\)-convex on \([a, b]\), for some fixed \( m ∈ (0,1] \) and \( q ∈ (1,∞)\), then
\[
\frac{f(a) + f(b)}{2} = \frac{1}{b-a} \int_a^b f(x)dx ≤ b-a \left( \frac{q-1}{2q-1} \right)^{\frac{q}{q-1}} \left( \mu_1^{\frac{1}{q}} + \mu_2^{\frac{1}{q}} \right) (1.4)
\]
where
\[
\mu_1 = \min \left\{ \frac{|f'(a)|^q + m|f'(\frac{a+b}{2})|^q}{2} , \frac{|f'(b)|^q + m|f'(\frac{a+b}{2})|^q}{2} \right\},
\]
\[
\mu_2 = \min \left\{ \frac{|f'(a)|^q + m|f'(\frac{a+b}{2})|^q}{2} , \frac{|f'(b)|^q + m|f'(\frac{a+b}{2})|^q}{2} \right\}.
\]

Let \( f : I \to R \) be a differentiable mapping on \( I \) such that \( f' ∈ L[a, b] \), where \(0 ≤ a < b < ∞\). If \( |f'|^q \) is \((α,m)\)-convex on \([a, b]\), for some fixed \( α, m ∈ (0,1] \) and \( q ∈ [1,∞)\), then
\[
\frac{f(a) + f(b)}{2} = \frac{1}{b-a} \int_a^b f(x)dx ≤ b-a \left( \frac{1}{2} \right)^{\frac{1}{q}} \times \min \left\{ \frac{v_1|f'(a)|^q + m\nu_1|f'(\frac{b}{m})|^q}{2} , \frac{v_1|f'(b)|^q + m\nu_2|f'(\frac{a}{m})|^q}{2} \right\},
\]
where
\[
v_1 = \frac{1}{(α+1)(α+2)} \left[ α + \frac{1}{2} \right]^a
\]
and
\[
v_2 = \frac{1}{(α+1)(α+2)} \left[ α^2 + α + 2 - \frac{1}{2} \right]^a.
\]

The main aim of this paper is to establish new inequalities of Hermite-Hadamard type for the class of functions whose derivatives in absolutely value at certain powers are \((α,m)\)-convex.

### 2. Inequalities for Functions Whose Derivatives Are \((α,m)\)–Convex

In order to prove our main results we need the following lemma:

**Lemma 2.1** Let \( f : I ⊂ R \to R \) be a differentiable mapping on \( I' \), \( a, b ∈ I \) with \( a < b \). If \( f' ∈ L[a, b] \) and \( λ, μ ∈ [0,∞), λ + μ > 0 \), then

the following equality holds:

**Theorem 1.4.** Let \( I \) be an open real interval such
\[
\frac{\lambda f(a) + \mu f(b)}{\lambda + \mu} - \frac{1}{b-a} \int_{a}^{b} f(x)dx = \frac{b-a}{\lambda + \mu} \left[(\lambda + \mu)t - \lambda\right] f'(tb + (1-t)a)dt
\]

Proof. Integration by parts we have

\[
I = \int_{0}^{1} \left[(\lambda + \mu)t - \lambda\right] f'(tb + (1-t)a)dt
\]

\[
= \left[\frac{b-a}{\lambda + \mu} f(tb + (1-t)a)\right]_{0}^{1} - \frac{\lambda + \mu}{b-a} \int_{0}^{1} f(tb + (1-t)a)dt.
\]

Setting \( x = tb + (1-t)a \), and \( dx = (b-a)dt \) gives

\[
I = \frac{\lambda f(a) + \mu f(b)}{b-a} - \frac{\lambda + \mu}{(b-a)^2} \int_{a}^{b} f(x)dx.
\]

Therefore,

\[
\left(\frac{b-a}{\lambda + \mu}\right) I = \frac{\lambda f(a) + \mu f(b)}{\lambda + \mu} - \frac{1}{b-a} \int_{a}^{b} f(x)dx
\]

which completes the proof. \( \square \)

The next theorem gives a new refinement of the upper Hermite-Hadamard inequality for \((\alpha, m)\)-convex functions.

**Theorem 2.2.** Let \( f : I \subset [0, \infty) \rightarrow \mathbb{R} \) be a differentiable mapping on \( I' \) such that \( f' \in L[a,b] \), where \( a,b \in I' \) with \( a < b \). If \( |f'|^q \) is \((\alpha, m)\)-convex on \([a, b]\), for some fixed \((\alpha, m) \in (0,1]^2, \lambda, \mu \in [0, \infty)\) with \( \lambda + \mu > 0 \), and \( q \geq 1 \), then the following inequality holds:

\[
\left| \frac{\lambda f(a) + \mu f(b)}{\lambda + \mu} - \frac{1}{b-a} \int_{a}^{b} f(x)dx \right| \leq \frac{b-a}{\lambda + \mu} \left[(\lambda + \mu)t - \lambda\right] f'(tb + (1-t)a)dt
\]

where

\[
\gamma_{1} = \frac{1}{(\alpha+1)(\alpha + 2)} \left[ \frac{2\lambda^{\alpha+2}}{(\lambda + \mu)^{\alpha+1}} + (\alpha+1)\lambda - \mu \right],
\]

\[
\gamma_{2} = \frac{\lambda^{2} + \mu^{2}}{2(\lambda + \mu)} - \gamma_{1},
\]

and

\[
\gamma_{3} = \frac{1}{(\alpha+1)(\alpha + 2)} \left[ \frac{2\mu^{\alpha+2}}{(\lambda + \mu)^{\alpha+1}} + (\alpha+1)\lambda - \mu \right],
\]

\[
\gamma_{4} = \frac{\lambda^{2} + \mu^{2}}{2(\lambda + \mu)} - \gamma_{3}.
\]

**Proof.** Suppose that \( q = 1 \). From Lemma 2.1 and using the \((\alpha, m)\)-convexity of \(|f'|\), we have

\[
\left| \frac{\lambda f(a) + \mu f(b)}{\lambda + \mu} - \frac{1}{b-a} \int_{a}^{b} f(x)dx \right| \leq \frac{b-a}{\lambda + \mu} \left[(\lambda + \mu)t - \lambda\right] f'(tb + (1-t)a)dt
\]

\[
\times \left[ f'(b) + m(1-t^\alpha) \right] \left[ f'(a) \right] dt
\]
\[
\begin{align*}
&\frac{b-a}{\lambda+\mu} \int_0^1 \left( (\lambda+\mu)t - \lambda \right) t^\alpha \, dt + m(1-t^\alpha) \left| (\lambda+\mu)t - \lambda \right| f' \left( \frac{a}{m} \right) \, dt \\
&\quad \leq \frac{b-a}{\lambda+\mu} \left( \gamma_3 \left| f'(a) \right| + m\gamma_4 \left| f' \left( \frac{b}{m} \right) \right| \right),
\end{align*}
\]
where
\[
\gamma_3 = \frac{1}{(\alpha+1)(\alpha+2)} \left[ \frac{2\lambda^{\alpha+2}}{(\lambda+\mu)^{\alpha+1}} + (\alpha+1)\lambda \right]
\]
and
\[
\gamma_4 = \frac{\lambda^2 + \mu^2}{2(\lambda+\mu)} - \gamma_3.
\]
which completes the proof for this case.

Suppose now that \( q \in (1, \infty) \). From Lemma 2.1 and using the Hölder’s integral inequality, we have
\[
\int_0^1 \left( \lambda+\mu \right) t - \lambda \left| f' \left( tb + (1-t)a \right) \right| \, dt \\
\leq \frac{b-a}{\lambda+\mu} \left( \int_0^1 \left( \lambda+\mu \right) t - \lambda \, dt \right)^{\frac{\alpha}{\alpha+1}} \times \left( \int_0^1 \left( \lambda+\mu \right) t - \lambda \left| f' \left( tb + (1-t)a \right) \right| \, dt \right)^{\frac{\alpha+1}{\alpha}}.
\]
Since \( \left| f' \right|^{\eta} \) is \((\alpha, m)\)-convex on \([a, b]\), we know that for every \( t \in [0, 1] \)
\[
\left| f' \left( tb + m(1-t) \frac{a}{m} \right) \right|^{\eta} \\
\leq t^\alpha \left| f' \left( b \right) \right|^{\eta} + m(1-t^\alpha) \left| f' \left( \frac{a}{m} \right) \right|^{\eta}.
\]
From (2.2), (2.3), (2.4) and (2.5), we have
\[
\int_0^1 \left( \lambda+\mu \right) t - \lambda \left| f' \left( tb + (1-t)a \right) \right| \, dt \\
= \int_0^1 \left( \lambda+\mu \right) t - \mu \left| f' \left( ta + (1-t)b \right) \right| \, dt
\]
Analogously we obtain
\[
\left| \frac{\lambda f(a) + \mu f(b)}{\lambda + \mu} - \frac{1}{b-a} \int_a^b f(x) \, dx \right| \\
\leq \frac{b-a}{\lambda + \mu} \left( \frac{\lambda^2 + \mu^2}{2(\lambda + \mu)} \right)^{\frac{1}{2}} \left( \gamma_1 |f'(b)|^q + m \gamma_2 |f'(a)|^q \right)^{\frac{1}{q}}
\]

and analogously

\[
\left| \frac{\lambda f(a) + \mu f(b)}{\lambda + \mu} - \frac{1}{b-a} \int_a^b f(x) \, dx \right| \\
\leq \frac{b-a}{\lambda + \mu} \left( \frac{\lambda^2 + \mu^2}{2(\lambda + \mu)} \right)^{\frac{1}{2}} \left( \gamma_1 |f(a)|^q + m \gamma_2 |f(b)|^q \right)^{\frac{1}{q}}
\]

which completes the proof. \( \Box \)

**Corollary 2.3.** Suppose that all the assumptions of Theorem 2.2 are satisfied,

i) In the inequality (2.1), If we choose \( \lambda = \mu \), we obtain the inequality in (1.5).

ii) In the inequality (2.1) If we choose \( \lambda = \mu, \ m = 1, \ q = 1 \) and \( \alpha = 1 \) we obtain the inequality in (1.2).

iii) In the inequality (2.1) If we choose \( m = \alpha = 1 \) we have

\[
\left| \frac{\lambda f(a) + \mu f(b)}{\lambda + \mu} - \frac{1}{b-a} \int_a^b f(x) \, dx \right| \\
\leq \frac{b-a}{\lambda + \mu} \left( \frac{\lambda^2 + \mu^2}{2(\lambda + \mu)} \right)^{\frac{1}{2}} \times \min \left\{ \left( \gamma_1 |f'(b)|^q + \gamma_2 |f'(a)|^q \right)^{\frac{1}{q}}, \left( \gamma_1 |f'(a)|^q + \gamma_2 |f'(b)|^q \right)^{\frac{1}{q}} \right\}
\]

(2.6)

\[
\gamma_1 = \frac{1}{6} \left( \frac{2\lambda^3}{(\lambda + \mu)^2} + 2\mu - \lambda \right), \quad \gamma_2 = \frac{\lambda^2 + \mu^2}{2(\lambda + \mu)} - \gamma_1,
\]

and

\[
\gamma_3 = \frac{1}{6} \left[ \frac{2\mu^3}{(\lambda + \mu)^2} + 2\lambda - \mu \right], \quad \gamma_4 = \frac{\lambda^2 + \mu^2}{2(\lambda + \mu)} - \gamma_3.
\]

**Theorem 2.4.** Let \( f : I \subset [0, \infty) \rightarrow \mathbb{R} \) be a differentiable mapping on \( I^a \) such that \( f' \in L[a,b] \), where \( a,b \in I^a \) with \( a < b \). If \( |f'|^q \) is \((\alpha, m)\)-convex on \([a,b]\), for some fixed \((\alpha, m) \in (0,1]^2, \lambda, \mu \in [0, \infty) \) with \( \lambda + \mu > 0 \), and \( q > 1 \), then the following inequality holds:

\[
\left| \frac{\lambda f(a) + \mu f(b)}{\lambda + \mu} - \frac{1}{b-a} \int_a^b f(x) \, dx \right| \leq \frac{b-a}{(\lambda + \mu)^2} \times \left( \frac{1}{p+1} \right)^{\frac{1}{q}} \left[ \lambda^2 M_1^{\frac{1}{q}} + \mu^2 M_2^{\frac{1}{q}} \right]
\]

(2.7)

where

\[
M_1 = \min \left\{ |f'(a)|^q + \alpha m f' \left( \frac{\lambda f(a) + \mu f(b)}{m(\lambda + \mu)} \right)^q, \left| f'(a) \right|^q + \alpha m f' \left( \frac{\lambda f(a) + \mu f(b)}{m(\lambda + \mu)} \right)^q \right\}, \quad \frac{1}{p} + \frac{1}{q} = 1.
\]

**Proof.** From Lemma 2.1 and using the Hölder inequality, we have
where we use the fact that
\[
\int_{0}^{\frac{b-a}{\lambda+\mu}} \left[ (\lambda + \mu)t - \lambda \right]^p dt = \frac{\lambda^{p+1}}{(p+1)(\lambda + \mu)}
\]
and by Theorem 1.2 we get

\[
\int_{a}^{b} f(t) dt = \frac{\lambda + \mu}{\lambda} \int_{0}^{\frac{b-a}{\lambda+\mu}} f'(t) dt
\]

where
\[
\lambda + \mu = \int_{0}^{1} |f'(tb + (1-t)a)|^q dt
\]

\[
\lambda + \mu = \int_{0}^{\frac{b-a}{\lambda+\mu}} |f'(x)|^q dx
\]

which completes the proof. □

**Corollary 2.5.** Suppose that all the assumptions of Theorem 2.4 are satisfied, in this case:

i) In the inequality (2.7) if we choose \( \lambda = \mu \) and \( \alpha = 1 \) we obtain the inequality in (1.4).

ii) In the inequality (2.7) if we choose \( m = \alpha = 1 \) we have

\[
\left| \frac{\lambda f(a) + \mu f(b)}{\lambda + \mu} - \frac{1}{b-a} \int_{a}^{b} f(x) dx \right| \leq \frac{b-a}{(\lambda + \mu)^2} \left( \frac{1}{p+1} \right) \left[ \lambda^2 M_1^{\frac{1}{q}} + \mu^2 M_2^{\frac{1}{q}} \right]
\]
\[ M_1 = \frac{|f'(a)|^q + |f'(b)|^q}{2}, \quad M_2 = \frac{|f'(a)|^q + |f'(b)|^q}{2}. \]

**Theorem 2.6.** Let \( f : I \subset [0, \infty) \rightarrow R \) be a differentiable mapping on \( I \) such that \( f' \in L([a,b]) \), where \( a,b \in I \) with \( a < b \). If \( |f'|^q \) is \((\alpha, m)\)-convex on \([a,b] \), for some fixed \((\alpha, m) \in (0,1]^2, \lambda, \mu \in [0, \infty) \) with \( \lambda + \mu > 0 \), and \( q > 1 \), then the following inequality holds:

\[
\lambda f(a) + \mu f(b) - \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{b-a}{\lambda + \mu} \left( \frac{\lambda^{p+1} + \mu^{p+1}}{(p+1)(\lambda + \mu)} \right)^\frac{1}{p} \left( \frac{1}{\alpha + 1} \right)^\frac{1}{q} \min \{K_1, K_2\},
\]

where

\[
K_1 = |f'(b)|^q + m\alpha \left| f' \left( \frac{a}{m} \right) \right|^q, \\
K_2 = |f'(a)|^q + m\alpha \left| f' \left( \frac{b}{m} \right) \right|^q,
\]

and \( \frac{1}{p} + \frac{1}{q} = 1 \).

**Proof.** From Lemma 2.1 and using the Hölder’s integral inequality, we have

\[
\lambda f(a) + \mu f(b) - \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{1}{b-a} \left| \int_0^1 (\lambda + \mu) t - \lambda \right| f'(tb + (1-t)a)dt
\]

\[
\leq \frac{b-a}{\lambda + \mu} \left[ \int_0^1 (\lambda + \mu) t - \lambda \right] dt
\]

\[
\leq \frac{b-a}{\lambda + \mu} \left( \frac{1}{b-a} \int_0^1 (\lambda + \mu) t - \lambda \right) \left| f'(tb + (1-t)a) \right| dt
\]

\[
\leq \frac{b-a}{\lambda + \mu} \left( \frac{1}{b-a} \int_0^1 (\lambda + \mu) t - \lambda \right) \left| f'(tb + (1-t)a) \right| dt
\]

\[
\left( \frac{1}{p+1} \right) \left( \frac{1}{\alpha + 1} \right)^\frac{1}{q} \min \{K_1, K_2\},
\]

\[ (2.9) \]

**Corollary 2.7.** Suppose that all the assumptions of Theorem 2.6 are satisfied, in this case:

i) In the inequality (2.9) if we choose \( \lambda = \mu \), then the following inequality holds:

\[
\frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{b-a}{\lambda + \mu} \left( \frac{1}{\alpha + 1} \right)^\frac{1}{q} \min \{K_1, K_2\},
\]

\[ (2.10) \]

ii) In the inequality (2.9) if we choose \( m = \alpha = 1 \), we have

\[
\left| \lambda f(a) + \mu f(b) \right| - \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{b-a}{\lambda + \mu}
\]

\[
\left( \frac{1}{p+1} \right) \left( \frac{1}{\alpha + 1} \right)^\frac{1}{q} \min \{K_1, K_2\},
\]

**3. SOME APPLICATIONS FOR SPECIAL MEANS**
Let us recall the following special means of two nonnegative number \( a, b \) with \( b \geq a \) and \( \alpha \in [0,1] \):

i) The weighted arithmetic mean

\[
A_\alpha(a,b) := \alpha a + (1-\alpha)b, \ a,b \geq 0.
\]

ii) The unweighted arithmetic mean

\[
A(a,b) := \frac{a+b}{2}, \ a,b \geq 0.
\]

iii) The weighted harmonic mean

\[
H_\alpha(a,b) := \left( \frac{\alpha}{a} + \frac{1-\alpha}{b} \right)^{-1}, \ a,b > 0.
\]

iv) The unweighted harmonic mean

\[
H(a,b) := \frac{2ab}{a+b}, \ a,b > 0.
\]

v) The Logarithmic mean

\[
L(a,b) := \begin{cases} 
\frac{b-a}{\ln b - \ln a} & \text{if } a \neq b, \\ 
b & \text{if } a = b 
\end{cases}, \ a,b > 0.
\]

vi) The p-Logarithmic mean

\[
L_p(a,b) := \begin{cases} 
\left( \frac{b^{p+1} - a^{p+1}}{(p+1)(b-a)} \right)^{\frac{1}{p}} & \text{if } a \neq b, \\ 
b & \text{if } a = b 
\end{cases}, \ a,b > 0, \ p \in \mathbb{Z} \setminus \{-1,0\}.
\]

**Proposition 3.1** Let \( a,b \in R \) with \( 0 < a < b \) and \( n \in \mathbb{Z}, \ |n| \geq 2 \). Then, we have the following inequality:

\[
|A_{\frac{1}{n}}(a^n, b^n) - L_n^n(a,b)| \leq \frac{b-a}{\lambda + \mu} \left( \frac{\lambda^2 + \mu^2}{2(\lambda + \mu)} \right)^{\frac{1}{n}}.
\]

where

\[
\gamma_1 = \frac{1}{6} \left[ \frac{3\lambda^2}{(\lambda + \mu)^2} + 2\mu - \lambda \right], \quad \gamma_2 = \frac{\lambda^2 + \mu^2}{2(\lambda + \mu)} - \gamma_1,
\]

\[
\gamma_3 = \frac{1}{6} \left[ \frac{3\mu^2}{(\lambda + \mu)^2} + 2\lambda - \mu \right], \quad \gamma_4 = \frac{\lambda^2 + \mu^2}{2(\lambda + \mu)} - \gamma_3,
\]

\( \lambda, \mu \in [0,\infty) \) with \( \lambda + \mu > 0 \) and \( n \geq 1 \).

**Proof.** The assertion follows from inequality (2.6) in Corollary 2.3, for \( f : (0,\infty) \to R, f(x) = x^n \).

\[ \square \]

**Proposition 3.2** Let \( a,b \in R \) with \( 0 < a < b \) and \( n \in \mathbb{Z}, |n| \geq 2 \). Then, we have the following inequality:

\[
|A_{\frac{1}{n}}(a^n, b^n) - L_n^n(a,b)| \leq \frac{b-a}{\lambda + \mu} \left( \frac{1}{p+1} \right)^{\frac{1}{n}} |n| \left[ \lambda^2 M_1 + \mu^2 M_2 \right]^\frac{1}{n},
\]

where

\[
M_1 = A\left( a^{(n-1)q}, A_{\frac{1}{n}}^{(n-1)q}(b,a) \right),
\]

\[
M_2 = A\left( b^{(n-1)q}, A_{\frac{1}{n}}^{(n-1)q}(b,a) \right),
\]

\( \lambda, \mu \in [0,\infty) \) with \( \lambda + \mu > 0 \), and \( q > 1 \) with \( \frac{1}{p} + \frac{1}{q} = 1 \).
Proof. The assertion follows from inequality (2.8) in Corollary 2.5, for \( f : (0, \infty) \rightarrow R, f(x) = x^n \)

\[ \lambda, \mu \in [0, \infty) \] with \( \lambda + \mu > 0 \), and \( q > 1 \) with \( \frac{1}{p} + \frac{1}{q} = 1 \).

Proof. The assertion follows from inequality (2.10) in Corollary 2.7, for \( f : (0, \infty) \rightarrow R, f(x) = x^\frac{1}{a} \).

Proposition 3.4. Let \( a, b \in R \) with \( 0 < a < b \). Then we have the following inequality:

\[
H^{-1}_{\frac{1}{\lambda}}(a, b) - L^{-1}_{\frac{1}{\lambda}}(a, b) \\
\leq \frac{b-a}{\lambda + \mu} \left( \frac{\lambda^2 + \mu^2}{2(\lambda + \mu)} \right)^{\frac{1}{p}}
\times \min \left\{ \left( \gamma_1 \frac{1}{b^{\frac{1}{a^q}}} + \gamma_2 \frac{1}{a^{\frac{1}{b^q}}} \right) + \left( \gamma_3 \frac{1}{a^{\frac{1}{b^q}}} + \gamma_4 \frac{1}{b^{\frac{1}{a^q}}} \right) \right\}
\]

where

\[
\gamma_1 = \frac{1}{6} \left[ \frac{2\lambda^3}{(\lambda + \mu)^3} + 2\mu - \lambda \right], \gamma_2 = \frac{\lambda^2 + \mu^2}{2(\lambda + \mu)} - \gamma_1,
\]

\[
\gamma_3 = \frac{1}{6} \left[ \frac{2\mu^3}{(\lambda + \mu)^3} + 2\lambda - \mu \right], \gamma_4 = \frac{\lambda^2 + \mu^2}{2(\lambda + \mu)} - \gamma_3,
\]

\( \lambda, \mu \in [0, \infty) \) with \( \lambda + \mu > 0 \) and \( q \geq 1 \).

Proof. The assertion follows from inequality (2.6) in Corollary 2.3, for \( f : (0, \infty) \rightarrow R, f(x) = x^n \)

Proposition 3.5. Let \( a, b \in R \) with \( 0 < a < b \). Then we have the following inequality:

\[
H^{-1}_{\frac{1}{x^{p}}} \left( a^q, b^q \right) \\
\leq \frac{b-a}{(\lambda + \mu)^2} \left( \frac{1}{p+1} \right) \left[ A^\frac{1}{\lambda} \left( a^{(n-1)q}, b^{(n-1)q} \right) \right]
\]

where

\[
M_1 = H^{-1}_{\frac{1}{x^{p}}} \left( a^{2q}, a^{2q} \right) \left( b, a \right),
\]

\[
M_2 = H^{-1}_{\frac{1}{x^{p}}} \left( b^{2q}, a^{2q} \right) \left( b, a \right)
\]

\( \lambda, \mu \in [0, \infty) \) with \( \lambda + \mu > 0 \), and \( q > 1 \) with \( \frac{1}{p} + \frac{1}{q} = 1 \).

Proof. The assertion follows from inequality (2.8) in Corollary 2.5, for \( f : (0, \infty) \rightarrow R, f(x) = x^\frac{1}{a} \).

Proposition 3.6. Let \( a, b \in R \) with \( 0 < a < b \).

Then we have the following inequality:

\[
H^{-1}_{\frac{1}{x^{p}}} \left( a^q, b^q \right) \\
\leq \frac{b-a}{(\lambda + \mu)^2} \left( \frac{1}{p+1} \right) \left[ A^\frac{1}{\lambda} \left( a^{q}, b^{q} \right) \right]
\]

\( \lambda, \mu \in [0, \infty) \) with \( \lambda + \mu > 0 \), and \( q > 1 \) with \( \frac{1}{p} + \frac{1}{q} = 1 \).

Proof. The assertion follows from inequality (2.10) in Corollary 2.7, for \( f : (0, \infty) \rightarrow R, f(x) = x^\frac{1}{a} \).

4. REFERENCES


