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A NEW GENERALIZATION OF SOME INTEGRAL INEQUALITIES AND THEIR APPLICATIONS

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ABSTRACT

In this paper, a new identity for differentiable functions is derived. A consequence of the identity is that we can derive a general inequality containing all of the midpoint, trapezoid, and Simpson inequalities for functions whose derivatives in absolute value at certain power are convex. Some applications to special means of real numbers are also given.

KEYWORDS: *Convex function, Simpson's inequality, Hermite-Hadamard's inequality, midpoint inequality, trapezoid ineqaulity.*

1.INTRODUCTION

Let $f:I\subseteq \mathbb{R}\rightarrow \mathbb{R}$ be a convex function defined on the interval I of real numbers and $a,b\in I$ with $a<b$. The following inequality

$$f\left(\frac{a+b}{2}\right)\leq \frac{1}{b-a}\int_a^b f(x)dx\leq \frac{f(a)+f(b)}{2} \quad (1)$$

holds. This double inequality is known in the literature as Hermite-Hadamard integral inequality for convex functions. See [2-8], for the results of the generalization, improvement and extention of the famous integral inequality (1).

The following inequality is well known in the literature as Simpson's inequality .

Let $f:[a,b]\rightarrow \mathbb{R}$ be a four times continuously differentiable mapping on (a,b) and $\left\Vert f^{(4)}\right\Vert_{\infty}=\sup_{x\in(a,b)}\left\vert f^{(4)}(x)\right\vert<\infty$. Then the following inequality holds:

$$\left\vert \frac{1}{3}\left[\frac{f(a)+f(b)}{2}+2f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right\vert \leq \frac{1}{2880}\left\Vert f^{(4)}\right\Vert_{\infty}(b-a)^2 .$$

In recent years many authors have studied error estimations for Simpson's inequality; for refinements, counterparts, generalizations and new Simpson's type inequalities, see [1, 8-10]

In [10], Sarikaya et al. obtained inequalities for differentiable convex mapping which are connected Simpson's inequality, and they used the following lemma to prove this.

Lemma 1.1 *Let $f:I\subset \mathbb{R}\rightarrow \mathbb{R}$ be an absolutely continuous mapping on I° such that $f^{\prime}\in L[a,b]$, where $a,b\in I^{\circ}$ with $a<b$. Then the following equality holds:*

$$\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)dx$$

$$=\frac{b-a}{2}\int_0^1\left[\left(\frac{t}{2}-\frac{1}{3}\right)f'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)+\left(\frac{1}{3}-\frac{t}{2}\right)f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right]dt$$

The main inequality in [10], pointed out, is as follows.

Theorem 1.2 *Let* $f:I\subset\mathbb{R}\to\mathbb{R}$ *be a differentiable mapping on* I°, *such that* $f'\in L[a,b]$, *where* $a,b\in I^\circ$ *with* $a<b$. *If If* $|f'|^q$ *is convex on* $[a,b]$, $q>1$, *then the following inequality holds,*

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right|$$

$$\leq\frac{b-a}{12}\left(\frac{1+2^{p+1}}{3(p+1)}\right)^{\frac{1}{p}}\left\{\left(\frac{3|f'(b)|^q+|f'(a)|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{3|f'(a)|^q+|f'(b)|^q}{4}\right)^{\frac{1}{q}}\right\}, \qquad (2)$$

where $\frac{1}{p}+\frac{1}{q}=1.$

In [9], Sarikaya et al. obtained a new upper bound for the right-hand side of Simpson's inequality for convex mapping:

Corollary 1.3 *Let* $f:I\subset[0,\infty)\to\mathbb{R}$ *be a differentiable mapping on* I°, *such that* $f'\in L[a,b]$, *where* $a,b\in I^\circ$ *with* $a<b$. *If If* $|f'|^q$ *is convex on* $[a,b]$, $q>1$, *then the following inequality holds,*

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right|$$
(3)

$$\leq\frac{b-a}{12}\left(\frac{1+2^{p+1}}{3(p+1)}\right)^{\frac{1}{p}}\left\{\left(\frac{|f'(b)|^q+\left|f'\left(\frac{a+b}{2}\right)\right|^q}{2}\right)^{\frac{1}{q}}+\left(\frac{|f'(a)|^q+\left|f'\left(\frac{a+b}{2}\right)\right|^q}{2}\right)^{\frac{1}{q}}\right\},$$

where $\frac{1}{p}+\frac{1}{q}=1.$

In [6], some inequalities of Hermite-Hadamard type for differentiable convex mappings were presented as follows.

Theorem 1.4 *Let* $f:I\subset\mathbb{R}\to\mathbb{R}$ *be a differentiable mapping on* I°, $a,b\in I^\circ$ *with* $a<b$. *If If* $|f'|$ *is convex on* $[a,b]$, *then the following inequality holds,*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right|\leq\frac{b-a}{4}\left(\frac{|f'(a)|+|f'(b)|}{2}\right). \qquad (4)$$

In this paper, in order to provide a unified approach to midpoint inequality, trapezoid inequality and Simpson's inequality for functions whose derivatives in absolute value at certain power are convex, we derive a general integral identity for differentiable functions. Finally some applications for special means of real numbers are provided.

2. MAIN RESULTS

In order to prove our main theorems, we need the following Lemma.

Lemma 2.1 *Let* $f:I\subset\mathbb{R}\to\mathbb{R}$ *be a differentiable mapping on* I° *such that* $f'\in L[a,b]$, *where* $a,b\in I$ *with* $a<b$ *and* $\alpha,\lambda\in[0,1]$. *Then the following equality holds:*

$$\lambda\left(\alpha f(a)+(1-\alpha)f(b)\right)+(1-\lambda)f(\alpha a+(1-\alpha)b)-\frac{1}{b-a}\int_a^b f(x)dx \qquad (5)$$

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$$=(b-a)\left[\int_0^{1-\alpha}(t-\alpha\lambda)f'(tb+(1-t)a)dt\right.$$

$$\left.+\int_{1-\alpha}^{1}\left(t-1+\lambda(1-\alpha)\right)f'(tb+(1-t)a)dt\right].$$

Proof. We note that

$$I=\int_0^{1-\alpha}(t-\alpha\lambda)f'(tb+(1-t)a)dt$$

$$+\int_{1-\alpha}^{1}\left(t-1+\lambda(1-\alpha)\right)f'(tb+(1-t)a)dt$$

integrating by parts, we get

$$I=(t-\alpha\lambda)\frac{f(tb+(1-t)a)}{b-a}\bigg|_0^{1-\alpha}$$

$$-\int_0^{1-\alpha}\frac{f(tb+(1-t)a)}{b-a}dt$$

$$+\left(t-1+\lambda(1-\alpha)\right)\frac{f(tb+(1-t)a)}{b-a}\bigg|_{1-\alpha}^{1}$$

$$-\int_{1-\alpha}^{1}\frac{f(tb+(1-t)a)}{b-a}dt$$

$$=(1-\alpha-\alpha\lambda)\frac{f((1-\alpha)b+\alpha a)}{b-a}+\frac{\alpha\lambda f(a)}{b-a}$$

$$+\frac{(1-\alpha)\lambda f(b)}{b-a}$$

$$-\left(-\alpha+\lambda(1-\alpha)\right)\frac{f((1-\alpha)b+\alpha a)}{b-a}$$

$$-\int_0^1\frac{f(tb+(1-t)a)}{b-a}dt.$$

Setting $x=tb+(1-t)a$, and $dx=(b-a)dt$, we obtain

$$(b-a)I=\lambda\left(\alpha f(a)+(1-\alpha)f(b)\right)$$

$$+(1-\lambda)f(\alpha a+(1-\alpha)b)-\frac{1}{b-a}\int_a^b f(x)dx$$

which gives the desired representation (5).

Theorem 2.2 *Let $f:I\subset\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on I° such that $f'\in L[a,b]$, where $a,b\in I^\circ$ with $a<b$ and $\alpha,\lambda\in[0,1]$. If $|f'|^q$ is convex on $[a,b]$, $q\geq 1$, then the following inequality holds:*

$$\left|\lambda\left(\alpha f(a)+(1-\alpha)f(b)\right)+(1-\lambda)f(\alpha a+(1-\alpha)b)-\frac{1}{b-a}\int_a^b f(x)dx\right| \quad (6)$$

$$\leq\begin{cases}(b-a)\left\{\gamma_2^{1-\frac{1}{q}}\left(\mu_1|f'(b)|^q+\mu_2|f'(a)|^q\right)^{\frac{1}{q}}+\upsilon_2^{1-\frac{1}{q}}\left(\eta_3|f'(b)|^q+\eta_4|f'(a)|^q\right)^{\frac{1}{q}}\right\}, & \alpha\lambda\leq 1-\alpha\leq 1-\lambda(1-\alpha)\\ (b-a)\left\{\gamma_2^{1-\frac{1}{q}}\left(\mu_1|f'(b)|^q+\mu_2|f'(a)|^q\right)^{\frac{1}{q}}+\upsilon_1^{1-\frac{1}{q}}\left(\eta_1|f'(b)|^q+\eta_2|f'(a)|^q\right)^{\frac{1}{q}}\right\}, & \alpha\lambda\leq 1-\lambda(1-\alpha)\leq 1-\alpha\\ (b-a)\left\{\gamma_1^{1-\frac{1}{q}}\left(\mu_3|f'(b)|^q+\mu_4|f'(a)|^q\right)^{\frac{1}{q}}+\upsilon_2^{1-\frac{1}{q}}\left(\eta_3|f'(b)|^q+\eta_4|f'(a)|^q\right)^{\frac{1}{q}}\right\}, & 1-\alpha\leq\alpha\lambda\leq 1-\lambda(1-\alpha)\end{cases}$$

where

$$\gamma_1=(1-\alpha)\left[\alpha\lambda-\frac{(1-\alpha)}{2}\right],\ \gamma_2=(\alpha\lambda)^2-\gamma_1, \quad (7)$$

$$\upsilon_1=\frac{1-(1-\alpha)^2}{2}-\alpha\left[1-\lambda(1-\alpha)\right], \quad (8)$$

$$\upsilon_2=\frac{1+(1-\alpha)^2}{2}-(\lambda+1)(1-\alpha)\left[1-\lambda(1-\alpha)\right],$$

$$\mu_1=\frac{(\alpha\lambda)^3+(1-\alpha)^3}{3}-\alpha\lambda\frac{(1-\alpha)^2}{2}, \quad (9)$$

$$\mu_2 = \frac{1+\alpha^3+(1-\alpha\lambda)^3}{3} - \frac{(1-\alpha\lambda)}{2}(1+\alpha^2),$$

$$\mu_3 = \alpha\lambda\frac{(1-\alpha)^2}{2} - \frac{(1-\alpha)^3}{3},$$

$$\mu_4 = \frac{(\alpha\lambda-1)(1-\alpha^2)}{2} + \frac{1-\alpha^3}{3},$$

$$\eta_1 = \frac{1-(1-\alpha)^3}{3} - \frac{[1-\lambda(1-\alpha)]}{2}\alpha(2-\alpha), \quad (10)$$

$$\eta_2 = \frac{\lambda(1-\alpha)\alpha^2}{2} - \frac{\alpha^3}{3},$$

$$\eta_3 = \frac{[1-\lambda(1-\alpha)]^3}{3} - \frac{[1-\lambda(1-\alpha)]}{2}\left(1+(1-\alpha)^2\right) + \frac{1+(1-\alpha)^3}{3},$$

$$\eta_4 = \frac{[\lambda(1-\alpha)]^3}{3} - \frac{\lambda(1-\alpha)\alpha^2}{2} + \frac{\alpha^3}{3}.$$

Proof. Suppose that $q \geq 1$. From Lemma 2.1 and using the well known power mean inequality, we have

$$\left|\lambda(\alpha f(a)+(1-\alpha)f(b))+(1-\lambda)f(\alpha a+(1-\alpha)b) - \frac{1}{b-a}\int_a^b f(x)dx\right|$$

$$\leq (b-a)\left[\int_0^{1-\alpha}|t-\alpha\lambda|\left|f'(tb+(1-t)a)\right|dt \int_{1-\alpha}^{1}|t-1+\lambda(1-\alpha)|\left|f'(tb+(1-t)a)\right|dt\right]$$

$$\leq (b-a)\left\{\left(\int_0^{1-\alpha}|t-\alpha\lambda|dt\right)^{1-\frac{1}{q}} \times\left(\int_0^{1-\alpha}|t-\alpha\lambda|\left|f'(tb+(1-t)a)\right|^q dt\right)^{\frac{1}{q}} + \left(\int_{1-\alpha}^{1}|t-1+\lambda(1-\alpha)|dt\right)^{1-\frac{1}{q}} \times\left(\int_{1-\alpha}^{1}|t-1+\lambda(1-\alpha)|\left|f'(tb+(1-t)a)\right|^q dt\right)^{\frac{1}{q}}\right\}. \quad (11)$$

Since $|f'|^q$ is convex on $[a,b]$, we know that for $t\in[0,1]$

$$\left|f'(tb+(1-t)a)\right|^q \leq t\left|f'(b)\right|^q + (1-t)\left|f'(a)\right|^q,$$

hence, by simple computation

$$\int_0^{1-\alpha}|t-\alpha\lambda|dt = \begin{cases}\gamma_2, & \alpha\lambda\leq 1-\alpha \\ \gamma_1, & \alpha\lambda\geq 1-\alpha\end{cases}, \quad (12)$$

$$\gamma_1 = (1-\alpha)\left[\alpha\lambda - \frac{(1-\alpha)}{2}\right], \gamma_2 = (\alpha\lambda)^2 - \gamma_1,$$

$$\int_{1-\alpha}^{1}|t-1+\lambda(1-\alpha)|dt = \begin{cases}\upsilon_1, & 1-\lambda(1-\alpha)\leq 1-\alpha \\ \upsilon_2, & 1-\lambda(1-\alpha)\geq 1-\alpha\end{cases}, \quad (13)$$

$$\upsilon_1 = \frac{1-(1-\alpha)^2}{2} - \alpha[1-\lambda(1-\alpha)],$$

$$\upsilon_2 = \frac{1+(1-\alpha)^2}{2} - (\lambda+1)(1-\alpha)[1-\lambda(1-\alpha)],$$

$$\int_0^{1-\alpha}|t-\alpha\lambda|\left|f'(tb+(1-t)a)\right|^q dt \leq \int_0^{1-\alpha}|t-\alpha\lambda|\left[t\left|f'(b)\right|^q + (1-t)\left|f'(a)\right|^q\right]dt$$

$$= \begin{cases}\mu_1\left|f'(b)\right|^q + \mu_2\left|f'(a)\right|^q, & \alpha\lambda\leq 1-\alpha \\ \mu_3\left|f'(b)\right|^q + \mu_4\left|f'(a)\right|^q, & \alpha\lambda\geq 1-\alpha\end{cases}, \quad (14)$$

$$\mu_1 = \frac{(\alpha\lambda)^3+(1-\alpha)^3}{3} - \alpha\lambda\frac{(1-\alpha)^2}{2},$$

$$\mu_2 = \frac{1+\alpha^3+(1-\alpha\lambda)^3}{3} - \frac{(1-\alpha\lambda)}{2}(1+\alpha^2),$$

$$\mu_3 = \alpha\lambda\frac{(1-\alpha)^2}{2} - \frac{(1-\alpha)^3}{3},$$

$$\mu_4 = \frac{(\alpha\lambda - 1)(1-\alpha^2)}{2} + \frac{1-\alpha^3}{3},$$

and

$$\int_{1-\alpha}^{1} |t-1+\lambda(1-\alpha)| |f'(tb+(1-t)a)|^q dt$$

$$\leq \int_{1-\alpha}^{1} |t-1+\lambda(1-\alpha)| \left[t|f'(b)|^q + (1-t)|f'(a)|^q\right] dt$$

$$= \begin{cases} \eta_1 |f'(b)|^q + \eta_2 |f'(a)|^q, & 1-\lambda(1-\alpha) \leq 1-\alpha \\ \eta_3 |f'(b)|^q + \eta_4 |f'(a)|^q, & 1-\lambda(1-\alpha) \geq 1-\alpha \end{cases},$$

(15)

where η_1, η_2, η_3 and η_4 are defined as in (10). Thus, using (12)-(15) in (11), we obtain the inequality (6). This completes the proof.

Corollary 2.3 *Let the assumptions of Theorem 2.2 hold. Then for* $q=1$ *the inequality (6) reduced to the following inequality*

$$\left|\lambda(\alpha f(a) + (1-\alpha) f(b)) + (1-\lambda) f(\alpha a + (1-\alpha) b) - \frac{1}{b-a}\int_a^b f(x)dx\right|$$

(16)

$$\leq \begin{cases} (b-a)\left\{(\mu_1+\eta_3)|f'(b)| + (\mu_2+\eta_4)|f'(a)|\right\}, & \alpha\lambda \leq 1-\alpha \leq 1-\lambda(1-\alpha) \\ (b-a)\left\{(\mu_1+\eta_1)|f'(b)| + (\mu_2+\eta_2)|f'(a)|\right\}, & \alpha\lambda \leq 1-\lambda(1-\alpha) \leq 1-\alpha \\ (b-a)\left\{(\mu_3+\eta_3)|f'(b)| + (\mu_4+\eta_4)|f'(a)|\right\}, & 1-\alpha \leq \alpha\lambda \leq 1-\lambda(1-\alpha) \end{cases}$$

.

Corollary 2.4 *Let the assumptions of Theorem 2.2 hold. Then for* $\alpha = \frac{1}{2}$ *and* $\lambda = \frac{1}{3}$ *, from the inequality (6) we get the following Simpson type inequality*

$$\left|\frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{1}{b-a}\int_a^b f(x)dx\right|$$

(17)

$$\leq (b-a)\left(\frac{5}{72}\right)^{1-\frac{1}{q}} \left\{\left(\frac{29}{1296}|f'(b)|^q + \frac{61}{1296}|f'(a)|^q\right)^{\frac{1}{q}} + \left(\frac{61}{1296}|f'(b)|^q + \frac{29}{1296}|f'(a)|^q\right)^{\frac{1}{q}}\right\},$$

which is the same of the inequality in [9, Theorem 10] for $s=1$.

Corollary 2.5 *Let the assumptions of Theorem 2.2 hold. Then for* $\alpha = \frac{1}{2}$ *and* $\lambda = 0$*, from the inequality (6) we get the following midpoint inequality*

$$\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx\right| \tag{18}$$

$$\leq \frac{b-a}{8}\left\{\left(\frac{|f'(b)|^q + 2|f'(a)|^q}{3}\right)^{\frac{1}{q}} + \left(\frac{2|f'(b)|^q + |f'(a)|^q}{3}\right)^{\frac{1}{q}}\right\}$$

Corollary 2.6 *In Corollary 2.5, if* $q=1$*, then we have the following midpoint inequality*

$$\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{4}\left(\frac{|f'(a)| + |f'(b)|}{2}\right). \tag{19}$$

which is the same of the inequality (4).

Corollary 2.7 *Let the assumptions of Theorem 2.2 hold. Then for* $\alpha = \frac{1}{2}$ *and* $\lambda = 1$*, from the inequality (6) we get the following trapezoid inequality*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right|$$

$$\leq \frac{b-a}{8}\left\{\left(\frac{|f'(b)|^q + 5|f'(a)|^q}{6}\right)^{\frac{1}{q}} + \left(\frac{5|f'(b)|^q + |f'(a)|^q}{6}\right)^{\frac{1}{q}}\right\}$$

Using Lemma 2.1 we shall give another result for convex functions as follows.

Theorem 2.8 *Let* $f: I \subset \mathbb{R} \to \mathbb{R}$ *be a differentiable mapping on* I° *such that* $f' \in L[a,b]$*, where* $a,b \in I^\circ$ *with* $a<b$ *and* $\alpha, \lambda \in [0,1]$*. If* $|f'|^q$ *is*

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convex on $[a,b]$, $q>1$, *then the following inequality holds:*

$$\left|\lambda\left(\alpha f(a)+(1-\alpha)f(b)\right)+(1-\lambda)f(\alpha a+(1-\alpha)b)-\frac{1}{b-a}\int_a^b f(x)dx\right|\le(b-a)\left(\frac{1}{p+1}\right)^{\frac{1}{p}}$$

(20)

$$\times\begin{cases}\left[(1-\alpha)^{\frac{1}{q}}\varepsilon_1^{\frac{1}{p}}\delta_1^{\frac{1}{q}}+\alpha^{\frac{1}{q}}\varepsilon_3^{\frac{1}{p}}\delta_2^{\frac{1}{q}}\right], & \alpha\lambda\le1-\alpha\le1-\lambda(1-\alpha)\\ \left[(1-\alpha)^{\frac{1}{q}}\varepsilon_1^{\frac{1}{p}}\delta_1^{\frac{1}{q}}+\alpha^{\frac{1}{q}}\varepsilon_4^{\frac{1}{p}}\delta_2^{\frac{1}{q}}\right], & \alpha\lambda\le1-\lambda(1-\alpha)\le1-\alpha\\ \left[(1-\alpha)^{\frac{1}{q}}\varepsilon_2^{\frac{1}{p}}\delta_1^{\frac{1}{q}}+\alpha^{\frac{1}{q}}\varepsilon_3^{\frac{1}{p}}\delta_2^{\frac{1}{q}}\right], & 1-\alpha\le\alpha\lambda\le1-\lambda(1-\alpha)\end{cases}$$

where

$$\delta_1=\frac{\left|f'((1-\alpha)b+\alpha a)\right|^q+\left|f'(a)\right|^q}{2}, \quad (21)$$

$$\delta_2=\frac{\left|f'((1-\alpha)b+\alpha a)\right|^q+\left|f'(b)\right|^q}{2},$$

$$\varepsilon_1=(\alpha\lambda)^{p+1}+(1-\alpha-\alpha\lambda)^{p+1},$$

$$\varepsilon_2=(\alpha\lambda)^{p+1}-(\alpha\lambda-1+\alpha)^{p+1},$$

$$\varepsilon_3=\left[\lambda(1-\alpha)\right]^{p+1}+\left[\alpha-\lambda(1-\alpha)\right]^{p+1},$$

$$\varepsilon_4=\left[\lambda(1-\alpha)\right]^{p+1}-\left[\lambda(1-\alpha)-\alpha\right]^{p+1},$$

and $\frac{1}{p}+\frac{1}{q}=1.$

Proof. From Lemma 2.1 and by Hölder's integral inequality, we have

$$\left|\lambda\left(\alpha f(a)+(1-\alpha)f(b)\right)+(1-\lambda)f(\alpha a+(1-\alpha)b)-\frac{1}{b-a}\int_a^b f(x)dx\right|$$

$$\le(b-a)\left[\int_0^{1-\alpha}|t-\alpha\lambda|\left|f'(tb+(1-t)a)\right|dt+\int_{1-\alpha}^1|t-1+\lambda(1-\alpha)|\left|f'(tb+(1-t)a)\right|dt\right]$$

$$\le(b-a)\left\{\left(\int_0^{1-\alpha}|t-\alpha\lambda|^p dt\right)^{\frac{1}{p}}\left(\int_0^{1-\alpha}\left|f'(tb+(1-t)a)\right|^q dt\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\int_{1-\alpha}^1|t-1+\lambda(1-\alpha)|^p dt\right)^{\frac{1}{p}}\left(\int_{1-\alpha}^1\left|f'(tb+(1-t)a)\right|^q dt\right)^{\frac{1}{q}}\right\}.$$

(22)

Since $\left|f'\right|^q$ is convex on $[a,b]$, for $\alpha\in[0,1)$ by the inequality (1), we get

$$\int_0^{1-\alpha}\left|f'(tb+(1-t)a)\right|^q dt=(1-\alpha)\left[\frac{1}{(1-\alpha)(b-a)}\int_a^{(1-\alpha)b+\alpha a}\left|f'(x)\right|^q dx\right]\le(1-\alpha)\frac{\left|f'((1-\alpha)b+\alpha a)\right|^q+\left|f'(a)\right|^q}{2}. \quad (23)$$

The inequality (23) holds for $\alpha=1$ too. Similarly, for $\alpha\in(0,1]$ by the inequality (1), we have

$$\int_{1-\alpha}^1\left|f'(tb+(1-t)a)\right|^q dt=\alpha\left[\frac{1}{\alpha(b-a)}\int_{(1-\alpha)b+\alpha a}^b\left|f'(x)\right|^q dx\right]\le\alpha\frac{\left|f'((1-\alpha)b+\alpha a)\right|^q+\left|f'(b)\right|^q}{2}. \quad (24)$$

The inequality (24) holds for $\alpha=0$ too. By simple computation

$$\int_0^{1-\alpha}|t-\alpha\lambda|^p dt=\begin{cases}\frac{(\alpha\lambda)^{p+1}+(1-\alpha-\alpha\lambda)^{p+1}}{p+1}, & \alpha\lambda\le1-\alpha\\ \frac{(\alpha\lambda)^{p+1}-(\alpha\lambda-1+\alpha)^{p+1}}{p+1}, & \alpha\lambda\ge1-\alpha\end{cases}, \quad (25)$$

and

$$\int_{1-\alpha}^1|t-1+\lambda(1-\alpha)|^p dt=\begin{cases}\frac{\left[\lambda(1-\alpha)\right]^{p+1}+\left[\alpha-\lambda(1-\alpha)\right]^{p+1}}{p+1}, & 1-\alpha\le1-\lambda(1-\alpha)\\ \frac{\left[\lambda(1-\alpha)\right]^{p+1}-\left[\lambda(1-\alpha)-\alpha\right]^{p+1}}{p+1}, & 1-\alpha\ge1-\lambda(1-\alpha)\end{cases}, \quad (26)$$

thus, using (23)-(26) in (22), we obtain the inequality (20). This completes the proof.

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Corollary 2.9 *Let the assumptions of Theorem 2.8 hold. Then for* $\alpha=\frac{1}{2}$ *and* $\lambda=\frac{1}{3}$, *from the inequality (20) we get the following Simpson type inequality*

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right| \quad (27)$$

$$\le\frac{b-a}{12}\left(\frac{1+2^{p+1}}{3(p+1)}\right)^{\frac{1}{p}}\left\{\left(\frac{\left|f'\left(\frac{a+b}{2}\right)\right|^q+\left|f'(a)\right|^q}{2}\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\frac{\left|f'\left(\frac{a+b}{2}\right)\right|^q+\left|f'(b)\right|^q}{2}\right)^{\frac{1}{q}}\right\},$$

which is the same of the inequality (3).

Remark 2.10 *We note that if we use convexity of* $|f'|^q$ *in the inequality (27) then we obtain the inequality (2).*

Corollary 2.11 *Let the assumptions of Theorem 2.8 hold. Then for* $\alpha=\frac{1}{2}$ *and* $\lambda=0$, *from the inequality (20) we get the following midpoint inequality*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right|\le\frac{b-a}{4}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}$$

$$\times\left\{\left(\frac{\left|f'\left(\frac{a+b}{2}\right)\right|^q+\left|f'(a)\right|^q}{2}\right)^{\frac{1}{q}}+\left(\frac{\left|f'\left(\frac{a+b}{2}\right)\right|^q+\left|f'(b)\right|^q}{2}\right)^{\frac{1}{q}}\right\}.$$

Corollary 2.12 *Let the assumptions of Theorem 2.8 hold. Then for* $\alpha=\frac{1}{2}$ *and* $\lambda=1$, *from the inequality (20) we get the following trapezoid inequality*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|\le\frac{b-a}{4}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}$$

$$\times\left\{\left(\frac{\left|f'\left(\frac{a+b}{2}\right)\right|^q+\left|f'(a)\right|^q}{2}\right)^{\frac{1}{q}}+\left(\frac{\left|f'\left(\frac{a+b}{2}\right)\right|^q+\left|f'(b)\right|^q}{2}\right)^{\frac{1}{q}}\right\}.$$

Theorem 2.13 *Let* $f:I\subset\mathbb{R}\to\mathbb{R}$ *be a differentiable mapping on* I° *such that* $f'\in L[a,b]$, *where with* $a<b$ *and* $\alpha,\lambda\in[0,1]$. *If is convex on* $[a,b]$, $q>1$, *then the following inequality holds,*

$$\left|\lambda\left(\alpha f(a)+(1-\alpha)f(b)\right)+(1-\lambda)f(\alpha a+(1-\alpha)b)\right.$$

$$\left.-\frac{1}{b-a}\int_a^b f(x)dx\right|\le(b-a)$$

(28)

$$\times\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\begin{cases}\left[\varepsilon_1^{\frac{1}{p}}\delta_3^{\frac{1}{q}}+\varepsilon_3^{\frac{1}{p}}\delta_4^{\frac{1}{q}}\right], & \alpha\lambda\le1-\alpha\le1-\lambda(1-\alpha)\\ \left[\varepsilon_1^{\frac{1}{p}}\delta_3^{\frac{1}{q}}+\varepsilon_4^{\frac{1}{p}}\delta_4^{\frac{1}{q}}\right], & \alpha\lambda\le1-\lambda(1-\alpha)\le1-\alpha\\ \left[\varepsilon_2^{\frac{1}{p}}\delta_3^{\frac{1}{q}}+\varepsilon_3^{\frac{1}{p}}\delta_4^{\frac{1}{q}}\right], & 1-\alpha\le\alpha\lambda\le1-\lambda(1-\alpha)\end{cases}$$

, where

$$\delta_3=\frac{\left|f'(b)\right|^q(1-\alpha)^2+(1-\alpha^2)\left|f'(a)\right|^q dt}{2},$$

$$\delta_4=\frac{\left|f'(b)\right|^q\alpha(2-\alpha)+\alpha^2\left|f'(a)\right|^q dt}{2},$$

$$\varepsilon_1=(\alpha\lambda)^{p+1}+(1-\alpha-\alpha\lambda)^{p+1},$$

$$\varepsilon_2=(\alpha\lambda)^{p+1}-(\alpha\lambda-1+\alpha)^{p+1},$$

$$\varepsilon_3=\left[\lambda(1-\alpha)\right]^{p+1}+\left[\alpha-\lambda(1-\alpha)\right]^{p+1},$$

$$\varepsilon_4=\left[\lambda(1-\alpha)\right]^{p+1}-\left[\lambda(1-\alpha)-\alpha\right]^{p+1},$$

and $\frac{1}{p}+\frac{1}{q}=1$.

Proof. From Lemma 2.1 and by Hölder's integral inequality, we have the inequality (22). Since Since

$|f'|^q$ is convex on $[a,b]$, we know that for $t\in[0,1-\alpha]$ and $t\in[1-\alpha,1]$

$$\left|f'(tb+(1-t)a)\right|^q \le t\left|f'(b)\right|^q+(1-t)\left|f'(a)\right|^q.$$

Hence

$$\Big|\lambda\big(\alpha f(a)+(1-\alpha)f(b)\big)+(1-\lambda)f(\alpha a+(1-\alpha)b)-\frac{1}{b-a}\int_a^b f(x)dx\Big|\le(b-a)$$
$$\times\left\{\left(\int_0^{1-\alpha}|t-\alpha\lambda|^p dt\right)^{\frac{1}{p}}\left(\int_0^{1-\alpha} t\left|f'(b)\right|^q+(1-t)\left|f'(a)\right|^q dt\right)^{\frac{1}{q}}\right.$$
$$+\left(\int_{1-\alpha}^{1}\left|t-1+\lambda(1-\alpha)\right|^p dt\right)^{\frac{1}{p}}$$
$$\left.\times\left(\int_{1-\alpha}^{1} t\left|f'(b)\right|^q+(1-t)\left|f'(a)\right|^q dt\right)^{\frac{1}{q}}\right\}$$

$$\le(b-a)\left\{\left(\int_0^{1-\alpha}|t-\alpha\lambda|^p dt\right)^{\frac{1}{p}}\right.$$
$$\times\left(\frac{\left|f'(b)\right|^q(1-\alpha)^2+(1-\alpha^2)\left|f'(a)\right|^q dt}{2}\right)^{\frac{1}{q}}$$
$$+\left(\int_{1-\alpha}^{1}\left|t-1+\lambda(1-\alpha)\right|^p dt\right)^{\frac{1}{p}} \qquad (29)$$
$$\left.\times\left(\frac{\left|f'(b)\right|^q\alpha(2-\alpha)+\alpha^2\left|f'(a)\right|^q dt}{2}\right)^{\frac{1}{q}}\right\}.$$

thus, using (25) and (26) in (29), we obtain the inequality (28). This completes the proof.

Corollary 2.14 *Let the assumptions of Theorem 2.13 hold. Then for* $\alpha=\frac{1}{2}$ *and* $\lambda=\frac{1}{3}$, *from the inequality (20) we get the following Simpson type inequality*

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\le\frac{b-a}{12}\left(\frac{1+2^{p+1}}{3(p+1)}\right)^{\frac{1}{p}}$$
$$\times\left\{\left(\frac{3\left|f'(b)\right|^q+\left|f'(a)\right|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{3\left|f'(a)\right|^q+\left|f'(b)\right|^q}{4}\right)^{\frac{1}{q}}\right\},$$

which is the same of the inequality (2).

3. SOME APPLICATIONS FOR SPECIAL MEANS

Let us recall the following special means of the two nonnegative numbers a and b with $\alpha\in[0,1]$:

1. The weighted arithmetic mean

$$A_\alpha(a,b):=\alpha a+(1-\alpha)b,\ a,b\ge0.$$

2. The unweighted arithmetic mean

$$A(a,b):=\frac{a+b}{2},\ a,b\ge0.$$

3. The weighted geometric mean

$$G_\alpha(a,b)=a^\alpha b^{1-\alpha},\ a,b>0.$$

4. The unweighted geometric mean

$$G(a,b)=\sqrt{ab},\ a,b>0.$$

5. The weighted harmonic mean

$$H_\alpha(a,b):=\left(\frac{\alpha}{a}+\frac{1-\alpha}{b}\right)^{-1},\ a,b>0.$$

6. The unweighted harmonic mean

$$H(a,b):=\frac{2ab}{a+b},\ a,b>0.$$

7. The Logarithmic mean

$$L(a,b):=\frac{b-a}{\ln b-\ln a},\ a\ne b,\ a,b>0.$$

8. Then n-Logarithmic mean

$$L_n(a,b):=\left(\frac{b^{n+1}-a^{n+1}}{(n+1)(b-a)}\right)^{\frac{1}{n}},\ n\in\mathbb{Z}\setminus\{-1,0\},\ a,b>0,\ a\ne b.$$

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9. The identric mean

$$I(a,b)=\frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}},a,b>0,\,a\neq b.$$

Proposition 3.1 *Let* $a,b\in\mathrm{R}$ *with* $0<a<b$ *and* $n\in\mathrm{Z},|n|\geq 2$. *Then, for* $\alpha,\lambda\in[0,1]$ *and* $q\geq 1$, *we have the following inequality:*

$$\left|\lambda A_\alpha\left(a^n,b^n\right)+(1-\lambda)A_\alpha^n(a,b)-L_n^n(a,b)\right|$$
$$\leq(b-a)|n|$$
$$\times\begin{cases}\left\{\gamma_2^{1-\frac{1}{q}}\left(\mu_1 b^{(n-1)q}+\mu_2 a^{(n-1)q}\right)^{\frac{1}{q}}+\upsilon_2^{1-\frac{1}{q}}\left(\eta_3 b^{(n-1)q}+\eta_4 a^{(n-1)q}\right)^{\frac{1}{q}}\right\}, & \alpha\lambda\leq 1-\alpha\leq 1-\lambda(1-\alpha)\\ \left\{\gamma_2^{1-\frac{1}{q}}\left(\mu_1 b^{(n-1)q}+\mu_2 a^{(n-1)q}\right)^{\frac{1}{q}}+\upsilon_1^{1-\frac{1}{q}}\left(\eta_1 b^{(n-1)q}+\eta_2 a^{(n-1)q}\right)^{\frac{1}{q}}\right\}, & \alpha\lambda\leq 1-\lambda(1-\alpha)\leq 1-\alpha,\\ \left\{\gamma_1^{1-\frac{1}{q}}\left(\mu_3 b^{(n-1)q}+\mu_4 a^{(n-1)q}\right)^{\frac{1}{q}}+\upsilon_2^{1-\frac{1}{q}}\left(\eta_3 b^{(n-1)q}+\eta_4 a^{(n-1)q}\right)^{\frac{1}{q}}\right\}, & 1-\alpha\leq\alpha\lambda\leq 1-\lambda(1-\alpha)\end{cases}$$

where $\gamma_1,\gamma_2,\upsilon_1,\ \upsilon_2,\ \ \mu_1,\mu_2,\mu_3,\mu_4,\eta_1,\eta_2,\eta_3,\eta_4$ numbers are defined as in (7)-(10).

Proof. The assertion follows from Theorem 2.2, for $f(x)=x^n,\,x\in[0,\infty),\,n\in\mathrm{Z},|n|\geq 2.$

Proposition 3.2 *Let* $a,b\in\mathrm{R}$ *with* $0<a<b$ *and* $n\in\mathrm{Z},|n|\geq 2$. *Then, for* $\alpha,\lambda\in[0,1]$ *and* $q>1$, *we have the following inequality:*

$$\left|\lambda A_\alpha\left(a^n,b^n\right)+(1-\lambda)A_\alpha^n(a,b)-L_n^n(a,b)\right|$$
$$\leq(b-a)\left(\frac{1}{p+1}\right)^{\frac{1}{p}}|n|$$
$$\times\begin{cases}\left[(1-\alpha)^{\frac{1}{q}}\varepsilon_1^{\frac{1}{p}}\theta_1+\alpha^{\frac{1}{q}}\varepsilon_3^{\frac{1}{p}}\theta_2\right], & \alpha\lambda\leq 1-\alpha\leq 1-\lambda(1-\alpha)\\ \left[(1-\alpha)^{\frac{1}{q}}\varepsilon_1^{\frac{1}{p}}\theta_1+\alpha^{\frac{1}{q}}\varepsilon_4^{\frac{1}{p}}\theta_2\right], & \alpha\lambda\leq 1-\lambda(1-\alpha)\leq 1-\alpha,\\ \left[(1-\alpha)^{\frac{1}{q}}\varepsilon_2^{\frac{1}{p}}\theta_1+\alpha^{\frac{1}{q}}\varepsilon_3^{\frac{1}{p}}\theta_2\right], & 1-\alpha\leq\alpha\lambda\leq 1-\lambda(1-\alpha)\end{cases}$$

where

$$\theta_1=A^{\frac{1}{q}}\left(A_\alpha^{(n-1)q}(a,b),a^{(n-1)q}\right),$$
$$\theta_2=A^{\frac{1}{q}}\left(A_\alpha^{(n-1)q}(a,b),b^{(n-1)q}\right),\ \frac{1}{p}+\frac{1}{q}=1,$$

and $\varepsilon_1,\,\varepsilon_2,\,\varepsilon_3,\,\varepsilon_4$ numbers are defined as in (21).

Proof. The assertion follows from Theorem 2.8, for $f(x)=x^n,\,x\in[0,\infty),\,n\in\mathrm{Z},|n|\geq 2.$

Proposition 3.3 *Let* $a,b\in\mathrm{R}$ *with* $0<a<b$. *Then, for* $\alpha,\lambda\in[0,1]$ *and* $q\geq 1$, *we have the following inequality:*

$$\left|\lambda H_\alpha^{-1}(a,b)+(1-\lambda)A_\alpha^{-1}(a,b)-L^{-1}(a,b)\right|$$

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$$\leq\begin{cases}(b-a)\left\{\gamma_2^{1-\frac{1}{q}}\left(\mu_1\frac{1}{b^{2q}}+\mu_2\frac{1}{a^{2q}}\right)^{\frac{1}{q}}+\upsilon_2^{1-\frac{1}{q}}\left(\eta_3\frac{1}{b^{2q}}+\eta_4\frac{1}{a^{2q}}\right)^{\frac{1}{q}}\right\}, & \alpha\lambda\leq 1-\alpha\leq 1-\lambda(1-\alpha)\\(b-a)\left\{\gamma_2^{1-\frac{1}{q}}\left(\mu_1\frac{1}{b^{2q}}+\mu_2\frac{1}{a^{2q}}\right)^{\frac{1}{q}}+\upsilon_1^{1-\frac{1}{q}}\left(\eta_1\frac{1}{b^{2q}}+\eta_2\frac{1}{a^{2q}}\right)^{\frac{1}{q}}\right\}, & \alpha\lambda\leq 1-\lambda(1-\alpha)\leq 1-\alpha,\\(b-a)\left\{\gamma_1^{1-\frac{1}{q}}\left(\mu_3\frac{1}{b^{2q}}+\mu_4\frac{1}{a^{2q}}\right)^{\frac{1}{q}}+\upsilon_2^{1-\frac{1}{q}}\left(\eta_3\frac{1}{b^{2q}}+\eta_4\frac{1}{a^{2q}}\right)^{\frac{1}{q}}\right\}, & 1-\alpha\leq\alpha\lambda\leq 1-\lambda(1-\alpha)\end{cases}$$

where $\gamma_1,\gamma_2,\upsilon_1,\ \upsilon_2,\ \ \mu_1,\mu_2,\mu_3,\mu_4,\eta_1,\eta_2,\eta_3,\eta_4$ numbers are defined as in (7)-(10).

Proof. The assertion follows from Theorem 2.2, for $f(x)=\frac{1}{x},\ x\in(0,\infty)$.

Proposition 3.4 *Let* $a,b\in\mathbb{R}$ *with* $0<a<b$. *Then, for* $\alpha,\lambda\in[0,1]$ *and* $q>1$, *we have the following inequality*

$$\left|\lambda H_\alpha^{-1}(a,b)+(1-\lambda)A_\alpha^{-1}(a,b)-L^{-1}(a,b)\right|$$

$$\leq(b-a)\left(\frac{1}{p+1}\right)^{\frac{1}{p}}$$

$$\times\begin{cases}\left[(1-\alpha)^{\frac{1}{q}}\varepsilon_1^{\frac{1}{p}}\theta_3+\alpha^{\frac{1}{q}}\varepsilon_3^{\frac{1}{p}}\theta_4\right], & \alpha\lambda\leq 1-\alpha\leq 1-\lambda(1-\alpha)\\\left[(1-\alpha)^{\frac{1}{q}}\varepsilon_1^{\frac{1}{p}}\theta_3+\alpha^{\frac{1}{q}}\varepsilon_4^{\frac{1}{p}}\theta_4\right], & \alpha\lambda\leq 1-\lambda(1-\alpha)\leq 1-\alpha,\\\left[(1-\alpha)^{\frac{1}{q}}\varepsilon_2^{\frac{1}{p}}\theta_3+\alpha^{\frac{1}{q}}\varepsilon_3^{\frac{1}{p}}\theta_4\right], & 1-\alpha\leq\alpha\lambda\leq 1-\lambda(1-\alpha)\end{cases}$$

where

$$\theta_3=H^{-\frac{1}{q}}\left(A_\alpha^{2q}(a,b),a^{2q}\right),$$

$$\theta_4=H^{-\frac{1}{q}}\left(A_\alpha^{2q}(a,b),b^{2q}\right),\ \frac{1}{p}+\frac{1}{q}=1,$$

and $\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4$ numbers are defined as in (21).

Proof. The assertion follows from Theorem 2.8, for $f(x)=\frac{1}{x},\ x\in(0,\infty)$.

Proposition 3.5 *Let* $a,b\in\mathbb{R}$ *with* $0<a<b$. *Then, for* $\alpha,\lambda\in[0,1]$ *and* $q\geq 1$, *we have the following inequality:*

$$\left|A_\lambda\left(\ln G_\alpha(a,b),\ln A_\alpha(a,b)\right)-\ln I(a,b)\right|$$

$$\leq\begin{cases}(b-a)\left\{\gamma_2^{1-\frac{1}{q}}\left(\mu_1\frac{1}{b^{q}}+\mu_2\frac{1}{a^{q}}\right)^{\frac{1}{q}}+\upsilon_2^{1-\frac{1}{q}}\left(\eta_3\frac{1}{b^{q}}+\eta_4\frac{1}{a^{q}}\right)^{\frac{1}{q}}\right\}, & \alpha\lambda\leq 1-\alpha\leq 1-\lambda(1-\alpha)\\(b-a)\left\{\gamma_2^{1-\frac{1}{q}}\left(\mu_1\frac{1}{b^{q}}+\mu_2\frac{1}{a^{q}}\right)^{\frac{1}{q}}+\upsilon_1^{1-\frac{1}{q}}\left(\eta_1\frac{1}{b^{q}}+\eta_2\frac{1}{a^{q}}\right)^{\frac{1}{q}}\right\}, & \alpha\lambda\leq 1-\lambda(1-\alpha)\leq 1-\alpha\\(b-a)\left\{\gamma_1^{1-\frac{1}{q}}\left(\mu_3\frac{1}{b^{q}}+\mu_4\frac{1}{a^{q}}\right)^{\frac{1}{q}}+\upsilon_2^{1-\frac{1}{q}}\left(\eta_3\frac{1}{b^{q}}+\eta_4\frac{1}{a^{q}}\right)^{\frac{1}{q}}\right\}, & 1-\alpha\leq\alpha\lambda\leq 1-\lambda(1-\alpha)\end{cases},$$

where $\gamma_1,\gamma_2,\upsilon_1,\ \upsilon_2,\ \ \mu_1,\mu_2,\mu_3,\mu_4,\eta_1,\eta_2,\eta_3,\eta_4$ numbers are defined as in (7)-(10).

Proof. The assertion follows from Theorem 2.2, for $f(x)=-\ln x,\ x>0$.

Proposition 3.6 *Let* $a,b\in\mathbb{R}$ *with* $0<a<b$. *Then, for* $\alpha,\lambda\in[0,1]$ *and* $q>1$, *we have the following inequality:*

$$\left|A_\lambda\left(\ln G_\alpha(a,b),\ln A_\alpha(a,b)\right)-\ln I(a,b)\right|$$

$$\leq(b-a)\left(\frac{1}{p+1}\right)^{\frac{1}{p}}$$

$$\times\begin{cases}\left[(1-\alpha)^{\frac{1}{q}}\varepsilon_1^{\frac{1}{p}}\theta_3+\alpha^{\frac{1}{q}}\varepsilon_3^{\frac{1}{p}}\theta_4\right], & \alpha\lambda\le 1-\alpha\le 1-\lambda(1-\alpha)\\ \left[(1-\alpha)^{\frac{1}{q}}\varepsilon_1^{\frac{1}{p}}\theta_3+\alpha^{\frac{1}{q}}\varepsilon_4^{\frac{1}{p}}\theta_4\right], & \alpha\lambda\le 1-\lambda(1-\alpha)\le 1-\alpha,\\ \left[(1-\alpha)^{\frac{1}{q}}\varepsilon_2^{\frac{1}{p}}\theta_3+\alpha^{\frac{1}{q}}\varepsilon_3^{\frac{1}{p}}\theta_4\right], & 1-\alpha\le\alpha\lambda\le 1-\lambda(1-\alpha)\end{cases}$$

where

$$\theta_3=H^{-\frac{1}{q}}\left(A_\alpha^q(a,b),a^q\right),$$

$$\theta_4=H^{-\frac{1}{q}}\left(A_\alpha^q(a,b),b^q\right),\ \frac{1}{p}+\frac{1}{q}=1,$$

and $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4$ numbers are defined as in (21).

Proof. The assertion follows from Theorem 2.8, for $f(x)=-\ln x,\ x>0.$

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