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A VARIATIONAL APPROACH TO STATIC ANALYSIS OF A THIN RECTANGULAR ORTHOTROPIC PLATE SUBJECTED TO UNIFORMLY DISTRIBUTED LOAD

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Abstract

In this paper, the values of numerical factors for deflection of a thin rectangular isotropic plate subjected to uniformly distributed load with different boundary conditions all round fixed, all round simply supported and two opposite sides fixed, and the other two simply supported are determined using variational approach. The results obtained using variation approach is compared with those obtained from literature. It is shown among other findings that variational solution does not yield a satisfactory result most especially for all round simply supported plates but produced satisfactory results for all round fixed and two opposite sides fixed and the other two simply supported plates

Keywords: numerical factors, rectangular orthotropic plate, boundary conditions, variational approach.

1.0 Introduction

A plate as an engineering structure is defined as a body in the shape of a prism with thickness, small in comparison with its other dimensions. They are commonly referred to as slabs or thin-walled structure. It is used in modern structures to transmit lateral and or in-plane load to adjacent support. It is one of the most important components employed in the main branches of engineering construction, building, civil engineering, hydraulic engineering, naval architecture and air-craft construction [Gould, 1999; Iyengar, 1988; Mansfield, 1989]. The classical method that leads to exact solution is not only rigorous and time consuming but proves in many cases quite laborious and almost impossible due to its mathematical difficulties [Charlton, 1961; Biot, 1972].The problems encountered in thin plate theory can be solved with the aid of various approximate methods such as energy method, finite element method, finite difference method and fourier series [Bares, 1969; Chajes, 1974; El Nachie, 1990]. However, common problems are encountered. For example, numerical methods (FDM and FEM) lead to an algebraic equation of large matrix size demanding large computer memories, thereby making the analysis cumbersome and time wasting. Navier's method is regarded as the widely used approximate method for the analysis of thin plates. Nevertheless, it is noted that the double trigonometric series in the method are not convenient for numerical computations if higher derivatives of the function "w" are involved. Besides, satisfactory solution by Navier method is only obtained for simply supported thin plates. This study highlights the use of polynomial function in the analysis thin rectangular orthotropic plate subjected to a centre point loading considering three different boundary conditions.

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3.0 Derivation of Governing Equations

The strain energy U of bending of plate is given by:

$$U = \frac{D}{2}\int_o^{lx}\int_o^{ly}\left\{\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)^2 - 2(1-\mu)\left[\left(\frac{\partial^2 w}{\partial x^2}\right)\left(\frac{\partial^2 w}{\partial y^2}\right)-\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2\right]\right\}dxdy \quad (1)$$

The potential energy of uniformly distributed load over the plate is given by:

$$V = -\int_0^{l_x}\int_0^{l_y} qw(x,y)dxdy \quad (2)$$

The total potential energy of the system, $\Pi_T = U - V$ (3)

Substitution of equations [1] and [2] into equation [3] gives

$$\Pi_T = \int_0^{l_x}\int_0^{l_y}\frac{D}{2}\left\{\left(\frac{\partial^2 w}{\partial w^2}\right)^2 + \left(\frac{\partial^2 w}{\partial y^2}\right)^2 - 2(1-\mu)\left[\frac{\partial^2 w}{\partial x^2}.\frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2\right]\right\}dxdy - \int_0^{l_x}\int_0^{l_y} qw(x,y)dxdy \quad (4)$$

Let

The coefficient A_i be considered as the coordinate defining the shape of the deflection surface.

l_x = length of plate in the x-direction

l_y = length of plate in the y-direction

ξ *and* η denotes the position of point load at any giving point in x and y coordinates respectively.

To evaluate equation [1], let the deflection $x(x, y)$ be given as

$$w(x, y) = A.\varphi(x).\psi(y) \quad (5)$$

Where

$\varphi(x)$ = Represent x –coordinate

$\varphi(y)$ = Represent y-coordinate

Let

$$\varphi(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 \quad \text{(a)}$$

$$\varphi(y) = b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 \quad \text{(b)} \quad (6)$$

By differentiating, equations [6(a)] and [6(b)], we have the following derivatives:

For x – direction:

$$\varphi(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4$$

$$\varphi^{/}(x) = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 \quad (7)$$

$$\varphi^{//}(x) = 2a_2 + 6a_3 x + 12a_4 x^2$$

$$\varphi^{///}(x) = 6a_3 + 24a_4 x$$

Likewise, for y – direction:

May 2013. Vol. 3, No. 3 ISSN2305-8269
International Journal of Engineering and Applied Sciences

www.eaas-journal.org

$$\psi(y)=b_0+b_1y+b_2y^2+b_3y^3+b_4y^4$$
$$\psi^{/}(y)=b_1+2b_2y+3b_3y^2+4b_4y^3$$
$$\psi^{//}(y)=2b_2+6b_3y+12b_4y^2 \qquad (8)$$
$$\psi^{///}(x)=6b_3+24b_4y$$

Determination of the coefficients using boundary conditions

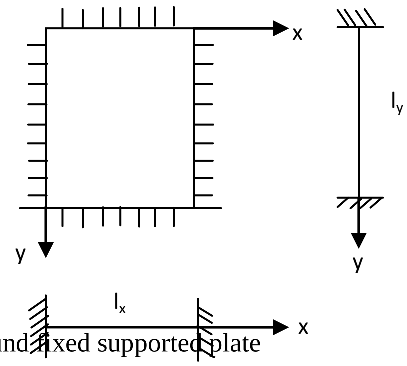

Figure 1: All round fixed supported plate

The boundary conditions are:

$$_{(0)}=0 \quad \varphi_{(l_x)}=0 \quad \varphi_{\left(\frac{l_x}{2}\right)}=1$$
$$\varphi^{/}{}_{(0)}=0 \quad \varphi^{/}{}_{(l_x)}=0 \qquad (9)$$

At x = 0:

$$\varphi_{(0)}=0 \quad =a_0 \qquad (a)$$
$$\varphi^{/}{}_{(0)}=0 \quad =a_1 \qquad (b) \qquad (10)$$

At x = l_x:

$$\varphi=0=a_0+a_1l_x+a_2l_x^2+a_3l_x^3+a_4l_x^4 \qquad (a)$$
$$\varphi^{/}=0=a_1+2a_2l_x+3a_3l_x^2+4a_4l_x^3 \qquad (b) \qquad (11)$$

(21)

At x = $\frac{l_x}{2}$:

$$\varphi=1=a_0+\frac{a_1l_x}{2}+\frac{a_2l_x^2}{4}+\frac{a_3l_x^3}{8}+\frac{a_4l_x^4}{16} \qquad (12)$$

Substituting [10 (a,b)] into [11(b)] gives

$$0=a_2l_x^2+a_3l_x^3+a_4l_x^4$$
$$a_2=-l_x(a_3+a_4l_x) \qquad (13)$$

Substituting equation [13] into [11(b)] gives

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$$0 = 0 - 2l_x^2\,(a_3 + a_4 l_x) + 3a_3 l_x^2 + 4a_4 l_x^3$$
$$a_3 = 2a_4 l_x \qquad (14)$$

Substituting equation [10 (a,b),13 and [14] into [12] gives

$$1 = 0 + 0 - \frac{l_x^3}{4}\left(-2a_4 l_x + a_4 l_x\right) - \frac{2a_4 l_x^4}{8} + \frac{a_4 l_x^4}{16}$$
$$1 = \frac{a_4 l_x^4}{2} - \frac{a_4 l_x^4}{4} - \frac{a_4 l_x^4}{4} + \frac{a_4 l_x^4}{16} = \frac{a_4 l_x^4}{16}$$
$$a_4 = \frac{16}{l_x^4} \qquad (15)$$

Substituting equation [15] into [14], gives

$$a_3 = \frac{2 \times 16 \times l_x}{l_x^4} = -\frac{32}{l_x^3} \qquad (16)$$

Substituting equation [15] and [16] into [13] gives

$$a_2 = -l_x\left(\frac{-32}{l_x^3} + \frac{16 l_x}{l_x^4}\right) = \frac{16}{l_x^2} \qquad (17)$$

Therefore,

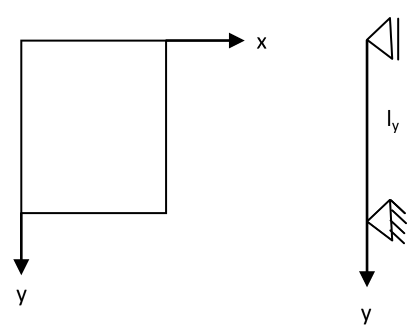
$$a_0 = 0,\quad a_1 = 0,\ a_2 = \frac{16}{l_x^2},\ a_3 = \frac{32}{l_x^3},\ a_4 = \frac{16}{l_x^4} \qquad (18)$$

Substituting equation [18] into [6] with respect to x *and* y respective we obtain

$$\varphi(x) = \frac{16x^2}{l_x^2} - \frac{32x^3}{l_x^3} + \frac{16x^4}{l_x^4} = 16\left[\frac{x^2}{l_x} - \frac{2x^3}{l_x^3} + \frac{x^4}{l_x^4}\right] \qquad (19)$$

Likewise

$$\psi(y) = \frac{16y^2}{l_y^2} - \frac{32y^3}{l_y^3} + \frac{16y^4}{l_y^4} = 16\left[\frac{y^2}{l_y^2} - \frac{2y^3}{l_y^3} + \frac{y^4}{l_y^4}\right] \qquad (20)$$



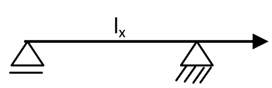


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Figure 2: All round simply supported plate.

The boundary conditions are:

$$\varphi_{(0)} = 0 \quad \varphi_{(l_x)} = 0 \quad \varphi_{\left(\frac{l_x}{2}\right)} = 1$$
$$\varphi^{//}{}_{(0)} = 0 \quad \varphi^{//}{}_{(l_x)} = 0 \tag{21}$$

At x = 0:

$$\varphi_{(0)} = 0 \quad = a_0 \qquad (a)$$
$$\varphi^{//}{}_{(0)} = 0 \quad = 2a_2 = a_2 \qquad (b) \tag{22}$$

At x = l_x:

$$\varphi_{(l_x)} = 0 = a_0 + a_1 l_x + a_2 l_x^2 + a_3 l_x^3 + a_4 l_x^4 \qquad (a)$$
$$\varphi^{//}{}_{(l_x)} = 0 = 2a_2 + 6a_3 l_x + 12 a_4 l_x^2 \qquad (b) \tag{23}$$

At x = $\frac{l_x}{2}$:

$$\varphi_{\left(\frac{l_x}{2}\right)} = 1 = a_0 + a_1 \frac{l_x}{2} + \frac{a_2 l_x^2}{4} + \frac{a_3 l_x^3}{8} + \frac{a_4 l_x^4}{16} \tag{24}$$

Substituting equation [22 (b)] into [23 (b)] gives:

$$0 = 6a_3 l_x + 12 a_4 l_x^2$$
$$a_3 = -2a_4 l_x \tag{25}$$

Substituting equation [22(a,b] and [25] into [23(a)] gives:

$$0 = 0 + a_1 l_x + 0 - 2a_4 l_x^4 + a_4 l_x^4$$
$$a_1 = a_4 l_x^3 \tag{26}$$

Substituting equation [22(a,b), 25 and 26] into [24] gives:

$$1 = 0 + \frac{a_4 l_x^4}{2} + 0 - \frac{2a_4 l_x^4}{8} + \frac{a_4 l_x^4}{16} = \frac{5a_4 l_x^4}{16} \tag{27}$$

$$a_4 = \frac{16}{5 l_x^4}$$

Substituting equation [27] into [25] gives:

$$a_3 = \frac{2 \times 16}{5 l_x^4} \times l_x = -\frac{32}{5 l_x^3} \tag{28}$$

Substituting equation [27] into [26] gives:

$$a_1 = \frac{16}{5 l_x^4} \times l_x^3 = \frac{16}{5 l_x} \tag{29}$$

ISSN2305-8269
International Journal of Engineering and Applied Sciences

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Therefore,

$$a_0 = 0,\ a_1 = \frac{16}{5l_x},\ a_2 = 0,\quad a_3 = -\frac{32}{5l_x^3},\ a_4 = \frac{16}{5l_x^4} \qquad (30)$$

Substituting equation [30] into [6(a)] gives:

$$\varphi(x) = \frac{16x}{5l_x} - \frac{32x^3}{5l_x^3} + \frac{16x^4}{l_x^4} \quad = \frac{16}{5}\left(\frac{x}{l_x} - \frac{2x^3}{l_x^3} + \frac{x^4}{l_x^4}\right) \qquad (31)$$

Similarly,

$$\psi(y) = \frac{16y}{5l_y} - \frac{32y^3}{5l_y^3} + \frac{16y^4}{5l_y^4} \quad = \frac{16}{5}\left(\frac{y}{l_y} - \frac{2y^3}{l_y^3} + \frac{y^4}{l_y^4}\right) \qquad (32)$$

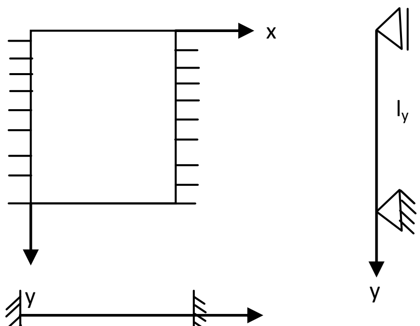

Figure 3: Two opposite sides fixed and the other two, simply supported plate

From equations, obtained for all around fixed and all round simply supported plates, we deduced that,

$$\varphi(x) = \frac{16x^2}{l_x^2} - \frac{32x^3}{l_x^3} + \frac{16x^4}{l4} = 16\left[\frac{x^2}{l_x^2} - \frac{2x^3}{l_x^3} + \frac{x^4}{l_x^4}\right] \qquad (33)$$

$$\psi(y) = \frac{16y}{5l_y} - \frac{32y^3}{5l_y^3} + \frac{16y^4}{5l_y^4} = \frac{16}{5}\left(\frac{y}{l_y} - \frac{2y^3}{l_y^3} + \frac{y^4}{l_y^4}\right) \qquad (34)$$

All round fixed rectangular plate:

Substituting equations [19] and [20] into [5] we obtain,

$$w(x,y) = A.16\left(\frac{x^2}{l_x^2} - \frac{2x^3}{l_x^3} + \frac{x^4}{l_x^4}\right).16\left(\frac{y^2}{l_y^2} - \frac{2y^3}{l_y^3} + \frac{y^4}{l_x^4}\right)$$

$$w(x,y) = A.256\left(\frac{x^2}{l_x^2} - \frac{2x^3}{l_x^3} + \frac{x^4}{l_x^4}\right)\left(\frac{y^2}{l_y^2} - \frac{2y^3}{l_y^3} + \frac{y^4}{l_x^4}\right) \qquad (35)$$

Hence,

$$\frac{\partial^2 w}{\partial x^2} = A.256\left(\frac{2}{l_x^2} - \frac{12x}{l_x^3} + \frac{12x^2}{l_x^4}\right)\left(\frac{y^2}{l_y^2} - \frac{2y^3}{l_y^3} + \frac{y^4}{l_x^4}\right) \qquad (36)$$

Similarly,

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International Journal of Engineering and Applied Sciences

www.eaas-journal.org

$$\frac{\partial^2 w}{\partial x^2} = A.256\left(\frac{2}{l_y^2} - \frac{12y}{l_y^3} + \frac{12y^2}{l_y^4}\right)\left(\frac{x^2}{l_x^2} - \frac{2x^3}{l_x^3} + \frac{x^4}{l_x^4}\right) \qquad (37)$$

Expanding equation [36, and 37] we have:

$$\frac{\partial^2 w}{\partial x^2} = A.256\left(\frac{2y^2}{l_x^2 l_y^2} - \frac{4y^3}{l_x^2 l_y^3} + \frac{2y^4}{l_x^2 l_y^4} - \frac{12xy^2}{l_x^3 l_y^2} + \frac{24xy^3}{l_x^3 l_y^3} - \frac{12xy^4}{l_x^3 l_y^4} + \frac{12x^2y^2}{l_x^4 l_y^2} - \frac{24x^2y^2}{l_x^4 l_y^3} + \frac{12x^2y^4}{l_x^4 l_y^4}\right) \qquad (38)$$

$$\frac{\partial^2 w}{\partial y^2} = A.256\left(\frac{2x^2}{l_x^2 l_y^2} - \frac{4x^3}{l_x^3 l_y^2} + \frac{2x^4}{l_x^4 l_y^2} - \frac{12x^2y}{l_x^2 l_y^3} + \frac{24x^3y}{l_x^3 l_y^3} - \frac{12x^4y}{l_x^4 l_y^3} + \frac{12x^2y^2}{l_x^2 l_y^4} - \frac{24x^3y^2}{l_x^4 l_y^4} + \frac{12x^4y^2}{l_x^4 l_y^4}\right) \qquad (39)$$

Hence,

$$\frac{\partial^2 w}{\partial x^2} = \left(A.256 \times \frac{2y^2}{l_x^2 l_y^2}\right)\left(1 - \frac{2y}{l_y} + \frac{y^2}{l_y^2} - \frac{6x}{l_x} + \frac{12xy}{l_x l_y} - \frac{6xy^2}{l_x l_y^2} + \frac{6x^2}{l_x^2} - \frac{12x^2y}{l_x^2 l_y} + \frac{6x^2y^2}{l_x^2 l_y^2}\right)$$

$$\frac{\partial^2 w}{\partial x^2} = \left(\frac{A512y^2}{l_x^2 l_y^2}\right)\left(1 - \frac{2y}{l_y} + \frac{y^2}{l_y^2} - \frac{6x}{l_x} + \frac{12xy}{l_x l_y} - \frac{6xy^2}{l_x l_y^2} + \frac{6x^2}{l_x^2} - \frac{12x^2y}{l_x^2 l_y} + \frac{6x^2y^2}{l_x^2 l_y^2}\right) \qquad (40)$$

$$\left(\frac{\partial^2 w}{\partial x^2}\right)^2 = A^2\left(\frac{512y^2}{l_x^2 l_y^2}\right)^2\left(1 - \frac{4y}{l_y} + \frac{6y^2}{l_y^2} - \frac{12x}{l_x} + \frac{48xy}{l_x l_y} - \frac{72xy^2}{l_x l_y^2} + \frac{48x^2}{l_x^2} - \frac{192x^2y}{l_x^2 l_y} + \frac{288x^2y^2}{l_x^2 l_y^2} - \frac{4y^3}{l_y^3} + \frac{48xy^3}{l_x l_y^3} - \frac{192x^2y^3}{l_x^3 l_y^3} + \frac{y^2}{l_y^4} - \frac{12xy^4}{l_x l_y^4} + \frac{48x^2y^4}{l_x^2 l_y^4} - \frac{72x^3}{l_x^3} + \frac{288x^3y}{l_x^3 l_y} - \frac{432x^3y^2}{l_x^3 l_y^2} + \frac{288x^3y^3}{l_x^3 l_y^3} - \frac{72x^3y^4}{l_x^3 l_y^4} + \frac{36x^4}{l_y^4} - \frac{144x^4y}{l_x^4 l_y} + \frac{216x^4y^2}{l_x^4 l_y^2} - \frac{144x^4y^3}{l_x^4 l_y^3} + \frac{36x^4y^4}{l_x^4 l_y^4}\right) \qquad (41)$$

Similarly,

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$$\left(\frac{\partial^2 W}{\partial y^2}\right)^2 = A^2\left(\frac{512x^2}{l_x^2 l_y^2}\right)^2\left(1-\frac{4x}{l_x}+\frac{6x^2}{l_x^2}-\frac{12y}{l_y}+\frac{48xy}{l_x l_y}-\frac{72x^2y}{l_x^2 l_y}+\frac{48y^2}{l_y^2}-\frac{192xy^2}{l_x l_y^2}\right.$$
$$+\frac{288x^2y^2}{l_x^2 l_y^2}-\frac{4x^3}{l_x^3}+\frac{48x^3y}{l_x^3 l_y}-\frac{192x^3y^2}{l_x^3 l_y^2}+\frac{x^4}{l_x^4}+\frac{12x^4y}{l_x^4 l_y}+\frac{48x^4y^2}{l_x^4 l_y^2}-\frac{72y^3}{l_y^3}$$
$$+\frac{288xy^3}{l_x l_y^3}-\frac{432x^2y^3}{l_x^2 l_y^3}+\frac{288x^3y^3}{l_x^3 l_y^3}-\frac{72x^4y^3}{l_x^4 l_y^3}+\frac{36y^4}{l_y^4}-\frac{144xy^4}{l_x l_y^4}+\frac{216x^2y^4}{l_x^2 l_y^4}$$
$$\left.-\frac{144x^3y^4}{l_x^3 l_y^4}+\frac{36x^4y^4}{l_x^4 l_y^4}\right) \tag{42}$$

According to Timoshenko and Woinowsky-Krieger (1959) for a polygonal plate if one of the boundary conditions is either w = 0 or $\partial w/\partial n$ = 0 where n = direction normal to the edge. The third term is negligible. Thus the strain energy equation (1)becomes :

$$U = \int_0^{l_x}\int_0^{l_y} \frac{D}{2}\left(\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + \left(\frac{\partial^2 w}{\partial y^2}\right)^2\right)dxdy \tag{43}$$

Hence, by substituting the derivatives of equations [41and 42] into equation [43], and integrating rigorously, we have:

$$U = \frac{D}{2}.A^2.\left(\frac{512}{l_x^2 l_y^2}\right)^2\left(\frac{7}{22050}\left(l_y^4 + l_x^4\right)\right) \tag{44}$$

Uniformly Distributed Load (UDL):

Substituting equations [35] into [2(a)] and integrating, we have:

$$V = -\frac{qA64l_x\, l_y}{225} \tag{45}$$

Substituting equations [44, and 46] into [3], and making the total potential energy a minimum, gives

$$\frac{\partial \Pi_T}{\partial A} = 0$$

$$0 = D.A.\left(\frac{512}{l_x^2\, l_y^2}\right)^2 \frac{7l_x l_y}{22050}\left(l_y^4 + l_x^4\right) - \frac{q64l_x l_y}{225}$$

Hence,

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$$A = \frac{0.003417968\, q l_x^4 l_y^4}{D\left(l_y^4 + l_x^4\right)} \tag{46}$$

Substituting equation [46]) into [5], we obtain:

$$w(x,y) = \frac{0.003418\, q l_x^4 l_y^4}{D\left(l_y^4 + l_x^4\right)} \varphi_{(x)} \psi_{(y)} \tag{47}$$

All round simply supported plate:

Substituting equation [31] and [32] into [5]gives:

$$w(x,y) = A.\frac{16}{5}\left(\frac{x}{l_x} - \frac{2x^3}{l_x^3} + \frac{x^4}{l_x^4}\right).\frac{16}{5}\left(\frac{y}{l_y} - \frac{2y^3}{l_y^3} + \frac{y^4}{l_y^4}\right)$$

$$= A = \frac{256}{25}\left(\frac{x}{l_x} - \frac{2x^3}{l_x^3} + \frac{x^4}{l_x^4}\right).\left(\frac{y}{l_y} - \frac{2y^3}{l_y^3} + \frac{y^4}{l_y^4}\right) \tag{48}$$

Therefore,

$$\frac{\partial w}{\partial x} = A\frac{256}{25}\left(\frac{l}{l_x} - \frac{6x^2}{l_x^3} + \frac{4x^3}{l_x^4}\right).\left(\frac{y}{l_y} - \frac{2y^3}{l_y^3} + \frac{y^4}{l_y^4}\right)$$

$$\frac{\partial^2 w}{\partial x^2} = A\frac{256}{25}\left(-\frac{12x}{l_x^3} + \frac{12x^2}{l_x^4}\right).\left(\frac{y}{l_y} - \frac{2y^3}{l_y^3} + \frac{y^4}{l_y^4}\right) \tag{49}$$

$$\frac{\partial^2 w}{\partial y^2} = A\frac{256}{25}\left(\frac{-12y}{l_y^3} + \frac{12y^2}{l_y^4}\right).\left(\frac{x}{l_x} - \frac{2x^3}{l_x^3} + \frac{x^4}{l_x^4}\right) \tag{50}$$

Expanding equation [49] we have:

$$\frac{\partial^2 w}{\partial x^2} = A\frac{256}{25}\left(\frac{-12xy}{l_y^3 l_y} + \frac{24xy^3}{l_x^3 l_y^3} - \frac{12xy^4}{l_x^3 l_x^4} + \frac{12x^2 y}{l_x^3 l_x} - \frac{24x^2 y^3}{l_x^4 l_y^3} + \frac{12x^2 y^4}{l_x^4 l_y^4}\right)$$

$$\frac{\partial^2 w}{\partial x^2} = A\frac{3072xy}{25 l_x l_y}\left(-\frac{1}{l_y^2} + \frac{2y^2}{l_x^2 l_y^2} - \frac{y^3}{l_x^2 l_y^3} + \frac{x}{l_x^3} - \frac{2xy^2}{l_x^3 l_y^2} + \frac{xy^3}{l_x^3 l_y^3}\right) \tag{51}$$

Hence,

International Journal of Engineering and Applied Sciences

www.eaas-journal.org

$$\left(\frac{\partial^2 w}{\partial x^2}\right)^2 = A^2\left(\frac{3072xy}{25l_xl_y}\right)^2\left(\frac{1}{l_x^4}-\frac{4y^2}{l_x^4l_y^2}+\frac{2y^3}{l_x^4l_y^3}-\frac{2x}{l_x^5}+\frac{8xy^2}{l_x^5l_y^2}-\frac{4xy^3}{l_x^5l_y^3}+\frac{4y^4}{l_x^4l_y^4}-\frac{4y^5}{l_x^4l_y^5}\right.$$

$$-\frac{8xy^4}{l_x^5l_y^4}+\frac{8xy^5}{l_x^5l_y^5}+\frac{y^6}{l_x^4l_y^6}-\frac{2xy^6}{l_x^5l_y^6}+\frac{x^2}{l_x^6}+\frac{4x^2y^2}{l_x^6l_y^2}+\frac{2x^2y^3}{l_x^6l_y^3}+\frac{4x^2y^4}{l_x^6l_y^4}$$

$$\left.-\frac{4x^2y^5}{l_x^6l_y^5}+\frac{x^2y^6}{l_x^6l_y^6}\right) \tag{52}$$

Similarly,

$$\left(\frac{\partial^2 w}{\partial y^2}\right)^2 = A^2\left(\frac{3072xy}{25l_xl_y}\right)^2\left(\frac{1}{l_y^4}-\frac{4x^2}{l_x^2l_y^4}+\frac{2x^3}{l_x^3l_y^4}-\frac{2y}{l_y^5}+\frac{8x^2y}{l_x^2l_y^5}-\frac{4x^3y}{l_x^3l_y^5}+\frac{4x^4}{l_x^4l_y^4}-\frac{4x^5}{l_x^5l_y^4}\right.$$

$$-\frac{8x^4y}{l_x^4l_y^5}+\frac{8x^5y}{l_x^5l_y^5}+\frac{x^6}{l_x^4l_y^6}-\frac{2x^6y}{l_x^5l_y^6}+\frac{x^2}{l_x^6}+\frac{4x^2y^2}{l_x^2l_y^6}+\frac{2x^3y^2}{l_x^3l_y^6}+\frac{4x^4y^2}{l_x^4l_y^4}$$

$$\left.-\frac{4x^5y^2}{l_x^5l_y^6}+\frac{x^6y^2}{l_x^6l_y^6}\right) \tag{53}$$

Substituting the derivatives of equations [52] and [53] into [43] and integrating, we obtain:

$$U=\frac{DA^2}{2}\left(\frac{3072}{25l_xl_y}\right)^2\left[\frac{31}{18900}\left(\frac{l_y^3}{l_x}+\frac{l_x^3}{l_y}\right)\right]=\frac{DA^2 12.383184}{l_x^2l_y^2}\left(\frac{l_y^3}{l_x}+\frac{l_x^3}{l_y}\right) \tag{54}$$

Substituting equation [48] into [2 (a)] and integrating, we obtain:

$$V=-\frac{qA256l_xl_y}{625} \tag{55}$$

Substituting equations [54 and 55] into [3], and making it a minimum we have:

$$\frac{\partial\Pi_T}{\partial A}=\Pi_T(A)=0$$

$$0=\frac{2DA12.383184}{l_x^2l_y^2}\left(\frac{l_y^3}{l_x}+\frac{l_x^3}{l_y}\right)-\frac{P256l_xl_y}{625}$$

May 2013. Vol. 3, No. 3 ISSN2305-8269
International Journal of Engineering and Applied Sciences

www.eaas-journal.org

$$A = \frac{q256 l_x^3 l_y^3}{D24.766368\left[\frac{l_y^3}{l_x} + \frac{l_x^3}{l_y}\right]625} \qquad (56)$$

Substituting equation [56] into [5], we obtain:

$$w(x,y) = \frac{0.016538557\, q l_x^2 l_y^3}{D\left[\frac{l_y^3}{l_x} + \frac{l_x^3}{l_y}\right]}\varphi(x)\psi(y) \qquad (57)$$

Two opposite sides fixed and the other side simply supported

Substituting equations [33 and 34] into [5] yields

$$\therefore w(x,y) = A\frac{16}{5}\left(\frac{x}{l_x} - \frac{2x^3}{l_x^3} + \frac{x^4}{l_x^4}\right)16\left(\frac{y^2}{l_y^2} - \frac{2y^3}{l_y^3} + \frac{y^4}{l_y^4}\right)$$

$$w(x,y) = A\frac{256}{5}\left(\frac{x}{l_x} - \frac{2x^3}{l_x^3} + \frac{x^4}{l_x^4}\right)\left(\frac{y^2}{l_y^2} - \frac{2y^3}{l_y^3} + \frac{y^4}{l_y^4}\right) \qquad (58)$$

$$\frac{\partial^2 w}{\partial x^2} = A\frac{256}{5}\left(-\frac{12x}{l_x^3} + \frac{12x^2}{l_x^4}\right)\left(\frac{y^2}{l_y^2} - \frac{2y^3}{l_y^3} + \frac{y^4}{l_y^4}\right) \qquad (59)$$

$$\frac{\partial^2 w}{\partial y^2} = A\frac{256}{5}\left(\frac{2}{l_y^2} - \frac{12y}{l_y^3} + \frac{12y^2}{l_y^4}\right)\left(\frac{x}{l_x} - \frac{2x^3}{l_x^3} + \frac{x^4}{l_x^4}\right) \qquad (60)$$

Expanding equations [59 and 60], we obtain:

$$\frac{\partial^2 w}{\partial x^2} = A\frac{256}{5}\left(\frac{12xy^2}{l_x^3 l_y^2} + \frac{24xy^3}{l_x^3 l_y^3} - \frac{12xy^4}{l_x^3 l_y^4} + \frac{12x^2y^2}{l_x^4 l_y^2} - \frac{24x^2y^3}{l_x^4 l_y^3} + \frac{12x^2y^4}{l_x^4 l_y^4}\right) \qquad (61)$$

$$\frac{\partial^2 w}{\partial y^2} = A\frac{256}{5}\left(\frac{2x}{l_x l_y^2} - \frac{4x^3}{l_x^3 l_y^2} - \frac{2x^4}{l_x^4 l_y^2} + \frac{12xy}{l_x l_y^3} + \frac{24x^3y}{l_x^3 l_y^3} - \frac{12x^4y}{l_x^4 l_y^3} + \frac{12xy^2}{l_x l_y^4} - \frac{24x^3y^2}{l_x^3 l_y^4} + \frac{12x^4y^2}{l_x^4 l_y^4}\right) \qquad (62)$$

ISSN2305-8269
International Journal of Engineering and Applied Sciences

www.eaas-journal.org

Hence,

$$\frac{\partial^2 w}{\partial x^2} = A\frac{256}{5}.\frac{2xy^2}{l_x^3 l_y^2}\left(-6+\frac{12y}{l_y}-\frac{6y^4}{l_y^2}+\frac{6x}{l_x}-\frac{12xy}{l_x l_y}+\frac{6xy^2}{l_x l_y^2}\right)$$

$$\left(\frac{\partial^2 w}{\partial x^2}\right)^2 = A^2\left(\frac{512xy^2}{5l_x^3 l_y^2}\right)^2\left(36-\frac{144y}{l_y}+\frac{216y^2}{l_y^2}-\frac{72x}{l_x}+\frac{288xy}{l_x l_y}-\frac{432xy^2}{l_x l_y^2}-\frac{144y^3}{l_y^3}+\frac{288xy^3}{l_x l_y^3}\right.$$

$$\left.+\frac{36y^4}{l_y^4}-\frac{72xy^4}{l_x l_y^4}+\frac{36x^2}{l_y^2}-\frac{144x^2y}{l_y^2 l_y}+\frac{216x^2y^2}{l_x^2 l_y^2}-\frac{144x^2y^3}{l_y^2 l_y^3}+\frac{36x^2y^4}{l_y^2 l_y^4}\right) \tag{63}$$

From equation [62], we obtain,

$$\left(\frac{\partial^2 w}{\partial x^2}\right) = A\frac{256\times 2x}{5l_x l_y^2}\left(1-\frac{2x^2}{l_x^2}+\frac{x^3}{l_x^3}-\frac{6y}{l_y}+\frac{12x^2y}{l_x^2 l_y}-\frac{6x^3y}{l_x^3 l_y}+\frac{6y^2}{l_y^2}-\frac{12x^2y^2}{l_x^2 l_y^2}+\frac{6x^3y^2}{l_x^2 l_y^4}\right)$$

$$\left(\frac{\partial^2 w}{\partial y^2}\right)^2 = A^2\left(\frac{512x}{5l_x l_y^2}\right)^2\left(1-\frac{4x^2}{l_x^2}+\frac{2x^3}{l_x^3}-\frac{12y}{l_y}+\frac{48x^2y}{l_x^2 l_y}-\frac{24x^3y}{l_x^3 l_y}+\frac{48y^2}{l_y^2}-\frac{192x^2y^2}{l_x^2 l_y^2}\right.$$

$$+\frac{96x^3y^2}{l_x^3 l_y^2}+\frac{4x^4}{l_x^4}-\frac{4x^5}{l_x^5}-\frac{48x^4y}{l_x^4 l_y}+\frac{48x^5y}{l_x^5 l_y}+\frac{192x^4y^2}{l_x^4 l_y^2}-\frac{192x^5y^2}{l_x^5 l_y^2}+\frac{x^6}{l_x^6}$$

$$-\frac{12x^6y}{l_x^6 l_y}+\frac{48x^6y^2}{l_x^6 l_y^2}-\frac{72y^3}{l_x^3}+\frac{288x^2y^3}{l_x^2 l_y^3}-\frac{144x^3y^3}{l_x^3 l_y^3}-\frac{288x^4y^3}{l_x^4 l_y^3}+\frac{288x^5y^3}{l_x^5 l_y^3}-\frac{72x^6y^3}{l_x^6 l_y^3}$$

$$\left.+\frac{36y^4}{l_y^4}-\frac{144x^2y^4}{l_x^2 l_y^4}+\frac{72x^3y^4}{l_x^3 l_y^4}+\frac{144x^4y^4}{l_x^4 l_y^4}-\frac{144x^5y^4}{l_x^5 l_y^4}+\frac{36x^6y^4}{l_x^6 l_y^4}\right) \tag{64}$$

Substituting equation [63 and 64] into [43] and integration, we obtain:

$$U = \frac{D}{2}A^2\left(\frac{512}{5l_x l_y}\right)^2\left\{\frac{1}{525}\left(\frac{l_y^3}{l_x}+\frac{31l_x^3}{6l_y}\right)\right\} \tag{65}$$

Uniformly Distributed Load (UDL):
Substituting equation [58] into [2(a)] and integrating we obtain

International Journal of Engineering and Applied Sciences

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$$V = -\frac{qA256l_x l_y}{750} \tag{66}$$

Substituting equations [65 and 66] into [3], and making it a minimum, we obtain:

$$\frac{\partial \Pi_p}{\partial A} = \partial \Pi_p(A) = 0$$

$$DA\left(\frac{512}{5l_x l_y}\right)^2 \frac{1}{525}\left(\frac{l_y^3}{l_x} + \frac{31l_x^3}{6l_y}\right) - \frac{q256l_x l_y}{750} = 0$$

Hence

$$A = \frac{0.000032552\,ql_x^3 l_y^3}{D\left\{\frac{1}{525}\left(\frac{l_y^3}{l_x} + \frac{31l_x^3}{6l_y}\right)\right\}} \tag{67}$$

Substituting equation [67] into [5], we have:

$$w(x,y) = \frac{0.000032552\,ql_x^3 l_y^3}{D\left\{\frac{1}{525}\left(\frac{l_y^3}{l_x} + \frac{31l_x^3}{6l_y}\right)\right\}}\varphi(x)\psi(y) \tag{68}$$

4.0 Results and Discussion

An example for numerical study

Table of results obtain for all round fixed, all round simply supported and two opposite sides fixed and the other two opposite side simply supported under transverse point and uniformly distributed load is presented below.

Taking the maximum deflection of the plate to be at the middle, for when the point load is applied at the middle and when it is entirely uniformly distributed as well. Then we have $\xi = x = {l_x}/{2}, \eta = y = {l_y}/{2}$.

May 2013. Vol. 3, No. 3 ISSN2305-8269
International Journal of Engineering and Applied Sciences

www.eaas-journal.org

Table 1: Table of values of numerical factor α for deflection of a uniformly distributed load on ARF, ASS, and 2opp(F&SS) rectangular plates for various values of span ratio l_y/l_x .

l_y/l_x	Energy method			Levy's method			Difference		
	ARF	ASS	2opp (F&SS)	ARF	ASS	2OPP(F&SS)	ARF	ASS	2opp (F&S.S)
1.0	$\frac{0.437499904q}{D}$	$\frac{2.116935296q}{D}$	$\frac{0.709457643q}{D}$	$\frac{0.32256q}{D}$	$\frac{1.03936q}{D}$	$\frac{0.49152q}{D}$	0.114939904(35.6)	1.077575296 (103.7)	0.217937643 (44.3)
1.1	$\frac{0.519900659q}{D}$	$\frac{2.515648689q}{D}$	$\frac{0.966015156q}{D}$	$\frac{0.3384q}{D}$	$\frac{1.2416q}{D}$	$\frac{0.6425q}{D}$	0.135900659 (35.4)	1.274048689 (102.6)	0.323455156 (50.3)
1.2	$\frac{0.590317413q}{D}$	$\frac{2.856374954q}{D}$	$\frac{1.25298316q}{D}$	$\frac{0.44032q}{D}$	$\frac{1.44384q}{D}$	$\frac{0.81664q}{D}$	0.149997413 (34.0)	1.412534954 (97)	0.436349315(53)
1.3	$\frac{0.6480866555q}{D}$	$\frac{3.135903581q}{D}$	$\frac{1.699501192q}{D}$	$\frac{0.48896q}{D}$	$\frac{1.63328q}{D}$	$\frac{0.99328q}{D}$	0.159126655 (32.5)	1.502623581 (92.0)	0.706221192 (71.2)
1.4	$\frac{0.6914274467q}{D}$	$\frac{3.359393024q}{D}$	$\frac{1.86572596q}{D}$	$\frac{0.52992q}{D}$	$\frac{1.8048q}{D}$	$\frac{1.1776q}{D}$	0.164354467 (30.5)	1.554593024(86.1)	0.68812596(58.4)
1.5	$\frac{0.730669942q}{D}$	$\frac{3.535500185q}{D}$	$\frac{1.966559893q}{D}$	$\frac{0.5632q}{D}$	$\frac{1.97632q}{D}$	$\frac{1.35936q}{D}$	0.167469942 (29.7)	1.559180185 (79.2)	0.607199893 (44.7)

May 2013. Vol. 3, No. 3 ISSN2305-8269
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www.eaas-journal.org

5.0 Discussion of Results

Looking at Table 1, it can be seen that the maximum coefficient value of deflection due to uniformly distributed load for all round fixed plate in terms of energy method is higher than the one obtained in Levy's method. It is observed that the percentage error is high for square plate (35.6%). The error diminishes as the span ratio increases, this indicates that the energy method converges faster as the span ratio increases, and increasing its number of terms would yield a better approximate result. The basic factor for this error is that the assumed polynomial function chosen is not so close to the actual deflection function (curve) as the Levy's single trigonometric function is.

In the case of all round simply supported plate in Table 1, no close agreement is noted for the maximum deflection coefficient values. The difference between the two methods are significantly outrageous, up to a maximum of 103.7%. However, the percentage error decreases as the span ratio increases. Nevertheless, it suggests that, the deflection function chosen does not suit energy method in terms of all round simply supported plate. Hence, cannot be recommended for use for simply supported plates.

In the case of plate with two opposite sides fixed and the other two sides simply supported in Table 1, the percentage error increases as the span ratio increases up to 1.3 and then begins to decline. From Table 1, it is observed that, the deflection coefficients decrease as the span ratio increase in all the three cases considered. Also, for two opposite sides fixed and the other two simply supported in Energy and Levy's methods respectively their results are closely the same.

6.0 CONCLUSION

The following conclusions are drawn from the study.

1. That the polynomial deflection function considered in this study does not yield a satisfactory result especially for all round simply supported plates. However, for all round fixed and two opposite sides fixed and the other two simply supported plates, it can be used. Because the results shown in Table 1 is for their first term, increasing the number of terms would yield a considerably upper bound approximate satisfactory result, though will lead to over design and consequently, uneconomical, yet durability and strength will be achieved.
2. The Navier's solution method and Levy's method stood better for solving simply supported plate problems especially in the light of the deflection function chosen.
3. The Energy method is best useful for in all round fixed plates.
4. Even though Energy method has been proved to be an excellent approximate method, this study has proved that unsatisfactory result can be obtained if an unsatisfactory deflection function is chosen.

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